Mark Scheme (Results)

Summer 2024

Pearson Edexcel GCE Mathematics

Pure 2 Paper 9MA0/02

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Summer 2024
Question Paper Log Number P75694A
Publications Code 9MA0_01_2406_MS

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 100.

2. The Edexcel Mathematics mark schemes use the following types of marks:
   - **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
   - **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
   - **B** marks are unconditional accuracy marks (independent of M marks)
   - Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol √ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- ⌐ The second mark is dependent on gaining the first mark

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses <u>and indicates which response they wish to submit</u>, examiners should mark this response.
   If there are several attempts at a question <u>which have not been crossed out</u>, examiners should mark the final answer which is the answer that is the <u>most complete</u>.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternatives answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used.

General Principles for Pure Mathematics Marking

(*But note that specific mark schemes may sometimes override these general principles*)

<u>Method mark for solving 3 term quadratic:</u>

1. Factorisation

$(x^2+bx+c)=(x+p)(x+q)$, where $|pq|=|c|$, leading to $x=\ldots$

$(ax^2+bx+c)=(mx+p)(nx+q)$, where $|pq|=|c|$ and $|mn|=|a|$, leading to $x=\ldots$

2. Formula

Attempt to use the correct formula (with values for *a*, *b* and *c*)

3. Completing the square

Solving $x^2+bx+c=0$: $\left(x\pm\frac{b}{2}\right)^2\pm q\pm c=0,\quad q\neq 0$, leading to $x=\ldots$

<u>Method marks for differentiation and integration:</u>

1. Differentiation

Power of at least one term decreased by 1. $(x^n\rightarrow x^{n-1})$

2. Integration

Power of at least one term increased by 1. $(x^n\rightarrow x^{n+1})$

<u>Use of a formula</u>

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

<u>Method mark</u> for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is <u>not</u> quoted, the method mark can be gained by implication from <u>correct</u> working with values but may be lost if there is any mistake in the working.

<u>Exact answers</u>

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Question	Scheme	Marks	AOs
1(a)(i)	$y=4x^3-7x^2+5x-10 \Rightarrow \left(\frac{dy}{dx}=\right)12x^2-14x+5$	M1 A1	1.1b 1.1b
(ii)	$\left(\frac{d^2y}{dx^2}=\right)24x-14$	A1ft	1.1b
		(3)	
(b)	$24x-14=0 \Rightarrow x=...$	M1	1.1b
	$x=\frac{7}{12}$ oe e.g. $x=\frac{14}{24}$	A1	1.1b
		(2)	
		(5 marks)	

Notes

(a)(i) If "+ c" is included with either derivative penalise it only once on the first occurrence.

M1: Award for $x^3 \to x^2$ or $x^2 \to x$ or $5x \to 5$ or $-10 \to 0$
Indices may be unprocessed e.g. $x^3 \to x^{3-1}$ or $x^2 \to x^{2-1}$ or $5x \to 5x^0$

A1: Correct <u>simplified</u> expression with indices processed $12x^2-14x+5$.
Do **not** allow x^1 for x or $5x^0$ for 5.
Apply isw if necessary once a correct answer is seen.
The "$\frac{dy}{dx}=$" is **not** required.

(ii)

A1ft: Correct simplified second derivative $24x-14$ or follow through their first derivative.
Must be <u>simplified</u> so do **not** allow e.g. x^1 for x or x^0 for 1 as above.
Apply isw if necessary once a correct answer is seen.
The "$\frac{d^2y}{dx^2}=$" is **not** required.

(b)

M1: Sets their **second derivative** of the form $ax+b,\ a,b\neq 0$ equal to 0 and proceeds to a value for x. Condone slips in rearranging as long as a value for x is obtained.
This may be implied by their value of x or may be implied by their working e.g.
$\left(\frac{d^2y}{dx^2}=\right)24x-14 \to 24x=14 \Rightarrow x=...$
Condone one slip in copying their second derivative.
Also condone if they "cancel" e.g. $\left(\frac{d^2y}{dx^2}=\right)24x-14 \to 12x-7=0 \Rightarrow x=...$

A1: Correct value from correct work and a correct second derivative but allow recovery if they "cancel" their second derivative to obtain e.g. $12x-7$.
Allow exact equivalents e.g. $\frac{14}{24}$ but not rounded decimals e.g. 0.583
Allow recurring decimal if clearly indicated e.g. $0.58\dot{3}$
Correct answer only from a correct second derivative (or correctly cancelled second derivative) scores both marks.
Isw after a correct answer is seen.

Question	Scheme	Marks	AOs
2(a)	$(u_{12} =)400+11\times-10=290$* or e.g. $(u_{12} =)400-110=290$* or e.g. $(u_{12} =)400+(12-1)\times-10=290$* or e.g. $(u_{12} =)410+12\times-10=290$*	B1*	1.1b
		(1)	
	Alternative 1:		
	$400+(n-1)\times-10=290$ $\Rightarrow 400-10n+10=290 \Rightarrow 10n=120 \Rightarrow n=12$*	B1*	1.1b
	Alternative 2:		
	$290=400+(12-1)d \Rightarrow 11d=-110 \Rightarrow d=-10$*	B1*	1.1b
(b)	$8100=\frac{1}{2}N(2\times400+(N-1)\times-10)$ or e.g. $8100=\frac{1}{2}N(400+400+(N-1)\times-10)$	M1	1.1b
	$8100=\frac{1}{2}N(2\times400+(N-1)\times-10)$ $\Rightarrow 16200=800N-10N^2+10N$ or e.g. $\Rightarrow 8100=400N-5N^2+5N$ $\Rightarrow N^2-81N+1620=0$*	A1*	2.1
		(2)	
(c)	$N^2-81N+1620=0 \Rightarrow (N-45)(N-36)=0 \Rightarrow N=45,36$	M1	1.1b
	$(N=)$ 36	A1	2.3
		(2)	
			(5 marks)

Notes

(a)

B1*: Correct working to obtain 290. Must be a correct calculation so do not condone missing brackets unless they are recovered. E.g. $(u_{12} =)400+12-1\times-10=290$ scores B0 unless followed by $=400+11\times-10=290$. Condone $(u_{12} =)400+(12-1)-10=290$

The "£" symbol is not required but the "290" must appear.

Alternative 1:

B1*: Correct working using the 290 to obtain n = 12.
There must be at least one intermediate line after setting up the equation and must be correct work so do not condone missing brackets unless they are recovered (as above).
A conclusion is **not** required with this approach as long as 12 is correctly obtained.

Alternative 2:

B1*: Correct working using the 290 and 400 to obtain $d = -10$.
There must be at least one intermediate line after setting up the equation and must be correct work so do not condone missing brackets unless they are recovered (as above).
A conclusion is **not** required with this approach as long as −10 is correctly obtained.

Allow candidates to list terms and show the 12th term is 290 e.g.

400, 390, 380, 370, 360, 350, 340, 330, 320, 310, 300, 290

Must list all 12 terms which must be correct and end with 290

Condone if missing 400, 390, 380 as these are given in the question.

(b) **Mark (b) and (c) together**

M1: Uses a correct sum formula in terms of N or n with $a = 400$ and $d = -10$ or $+10$ and sets = 8100. Condone e.g. > 8100 and allow A1 if this is recovered to become "=" before the final line.

Condone $8100 = \frac{1}{2}N\left(2\times400+\left(N-1\right)-10\right)$ if recovered or not.

A1*: Fully correct proof with sufficient working shown and no unrecovered errors.
Do not condone e.g. missing brackets or e.g. a missing N/n unless recovered before the final given answer.
Condone the use of n instead of N for **both** marks.
Condone terms in a different order as long as they are correct.
Condone $0 = N^2 - 81N + 1620*$
Sufficient working requires all brackets to be removed to obtain an unsimplified expanded quadratic before proceeding to the given answer including the "=0".

Alternative (further maths method): Series summation approach:

$$\sum_{r=1}^{N}\left(410-10r\right)=8100 \Rightarrow 410N-10\times\frac{1}{2}N\left(N+1\right)$$

$$\Rightarrow 410N-5N^2-5N=8100 \Rightarrow N^2-81N+1620=0*$$

M1: Attempt to sum an appropriate series with first term 400. Condone use of +10 as in the main scheme.

A1*: As main scheme.

(c)

M1: Solves the **given** quadratic equation by any correct method including a calculator to obtain at least one value for N. See general guidance for solving a 3-term quadratic.
If values are just written down and only one value is given it must be 45 or 36.
If both values are just written down they must both be correct.

A1: Realises that the smaller value is required and so selects (N =) 36.
Ignore any units if given.
The "N =" is not required, just look for the correct value.
It must be clear that this value has been selected. This may be indicated by e.g. underlining the 36 or the omission of the 45. If the 45 is not rejected score A0.
N = 36 with no working scores M1A1

Question	Scheme	Marks	AOs
3(i)	$(5,-2)$ **or e.g.** $x=5, y=-2$ o.e.	B1	1.1b
		(1)	
(ii)	$(1.5,-2)$ **or e.g.** $x=1.5, y=-2$ o.e.	B1	1.1b
		(1)	
(iii)	$(-3, \ldots)$ **or** $(\ldots, -1)$ **or** $x=-3$ **or** $y=-1$ o.e.	B1	1.1b
	$(-3,-1)$ **or** $x=-3$ **and** $y=-1$ o.e.	B1	1.1b
		(2)	
			(4 marks)

Notes

General guidelines for all parts:

Remember to check answers written against the questions.
If there is any contradiction, mark the answers given in the body of the script.
If there is no labelling, mark the responses in the order given.
The coordinates need to be values not just a calculation e.g. **not** $-2\times3+5$ for -1
Points can be written as a coordinate pair or separately as $x=\ldots, y=\ldots$
Do **not** allow coordinates written the wrong way round but isw if necessary
e.g. $x=5, y=-2\rightarrow(-2,5)$ scores B1 and isw

Condone missing brackets (one or both) e.g. 5, –2 **or** (5, –2 **or** 5, –2) for (5, –2)
Condone a missing comma e.g. $(5\ \ -2)$ for $(5,-2)$
Condone use of a semi-colon e.g. $(5\ ;-2)$ for $(5,-2)$
Condone vector notation e.g. $\begin{pmatrix}5\\-2\end{pmatrix}$ for $(5,-2)$ and condone $\left(\frac{5}{-2}\right)$

(i)

B1: $(5,-2)$ o.e. see above

(ii)

B1: $(1.5,-2)$ o.e. see above

(iii)

B1: One correct coordinate. See above.

B1: Both coordinates correct. See above.
Note that B0B1 is not a possible mark profile.
Note that in part (iii), some candidates show their thinking by transforming the point piecewise e.g. $(3,-2)\rightarrow(-3,-2)\rightarrow(-3,-6)\rightarrow(-3,-1)$
In such cases, mark their **final** pair of coordinates.

Question	Scheme	Marks	AOs
4(a)	$u_1 = 6 \Rightarrow u_2 = 6k - 5$ $u_2 = 6k - 5 \Rightarrow u_3 = k(6k-5) - 5$ $\Rightarrow k(6k-5) - 5 = -1$	M1	1.1b
	$\Rightarrow 6k^2 - 5k - 4 = 0$*	A1*	2.1
		(2)	
	Alternative:		
	$u_3 = -1 \Rightarrow -1 = ku_2 - 5 \Rightarrow u_2 = \frac{4}{k}$ $u_1 = 6 \Rightarrow u_2 = 6k - 5 \Rightarrow \frac{4}{k} = 6k - 5$	M1	1.1b
	$\Rightarrow 6k^2 - 5k - 4 = 0$*	A1*	2.1
(b)(i)	$k = \frac{4}{3}$	B1	2.2a
(ii)	$k = \frac{4}{3} \Rightarrow u_2 = \frac{4}{3} \times 6 - 5 \Rightarrow \sum_{r=1}^{3} u_r = 6 + \frac{4}{3} \times 6 - 5 - 1$	M1	1.1b
	$\sum_{r=1}^{3} u_r = 8$	A1	1.1b
		(3)	
			(5 marks)

Notes

(a)

M1: Correct application of the given recurrence relation using $u_1 = 6$ to find u_2 and then u_3 in terms of k and sets $u_3 = -1$

Condone missing brackets if the intention is clear e.g. $u_2 = 6k - 5 \Rightarrow u_3 = k\,6k - 5 - 5$

A1*: Obtains the printed answer with no errors including the "= 0"

This is a <u>given answer</u> so do not condone slips/missing brackets unless they are recovered **before** the final printed answer.

Alternative:

M1: Correct application of the given recurrence relation using $u_3 = -1$ to find u_2 in terms of k and then uses $u_1 = 6$ to find another expression for u_2 in terms of k and equates the 2 expressions.

A1*: Obtains the printed answer with no errors including the "= 0"

This is a <u>given answer</u> so do not condone slips unless they are recovered **before** the final printed answer.

(b)(i)

B1: Deduces the correct value of k. Ignore any working and just look for this value.

Allow equivalent exact values e.g. $1\frac{1}{3}$ or $1.\dot{3}$ but not clearly rounded e.g. 1.333

It must be clear that $k = \frac{4}{3}$ is selected so if both roots are offered score B0 unless $k = \frac{4}{3}$ is clearly intended by the calculation in part (ii)

(ii)

M1: Attempts the second term by e.g. $(\text{their } k)\times 6-5$ and then adds 6 and -1 to their second term. E.g. $6+"\frac{4}{3}"\times 6-5-1$

If they use u_1 and u_3 they must be as given in the question but condone a clear mis-copy of their u_2 value.

The attempt at the second term may be implied by their value.

Note that they may use $u_3=-1$ to find u_2 e.g. $-1="\frac{4}{3}"u_2-5 \Rightarrow u_2="\frac{3}{4}"(5-1)=3$

Condone slips when rearranging as long as the intention is clear.

The attempt at the second term may be seen embedded in their attempt at the sum e.g.

$\sum_{r=1}^{3} u_r = 6+\underline{\frac{4}{3}\times 6-5}-1$ or e.g. $\sum_{r=1}^{3} u_r = 6+\underline{\frac{3}{4}(5-1)}-1$

If they use both of their values for k allow M1.

Alternatives:

Note that $\sum_{r=1}^{3} u_r = 6+6k-5-1=6k$ so you may just see an attempt at $6k$ with their $\frac{4}{3}$.

Note that $\sum_{r=1}^{3} u_r = 6k^2+k-4$ so you may just see an attempt at $6k^2+k-4$ with their $\frac{4}{3}$.

A1: Correct value of 8 and no other values unless rejected.
Correct answer with no working scores both marks.
Allow recovery from an inexact value from part (i) e.g. 1.333

Question	Scheme	Marks	AOs
5	One of $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\frac{(3\theta)^2}{2}\right)$ or equivalents.	B1	1.1a
	$\dfrac{\theta\tan 2\theta}{1-\cos 3\theta} = \dfrac{\theta\times 2\theta}{1-\left(1-\frac{(3\theta)^2}{2}\right)}$	M1	2.1
	$=\dfrac{4}{9}$ or exact equivalent.	A1	1.1b
		(3)	
		(3 marks)	

Notes

B1: Award this mark for $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\dfrac{(3\theta)^2}{2}\right)$ or equivalents.

May be seen when working on numerator or denominator separately or within the fraction.

This is a B mark so if awarding for $\cos 3\theta$ do not condone missing brackets e.g. $1-\dfrac{3\theta^2}{2}$ unless they are recovered or are implied by subsequent work.

M1: Attempts to use both correct small angle approximations in the given expression.

For this mark they must have attempted to use $\tan 2\theta = 2\theta$ and $\cos 3\theta = 1-\dfrac{(3\theta)^2}{2}$ in the given expression but condone poor bracketing e.g. $\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}$ or e.g. $\dfrac{\theta\times 2\theta}{1-1-\frac{3\theta^2}{2}}$

Do not allow e.g. $\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$ as this suggests they are approximating $\dfrac{\theta\tan 2\theta}{\cos 3\theta}$

A1: Correct value. Do not allow rounded decimals e.g. 0.444 but allow if recurring decimals are clearly indicated e.g. $0.\dot{4}$ Do not allow e.g. $\dfrac{2}{4.5}$. Ignore any units if given.

Isw once a correct answer is seen.

<u>Examples:</u>

$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \dfrac{4}{3}\left(\text{or } -\dfrac{4}{3}\right)$	scores B1M1A0 (Missing brackets not recovered)
$\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$	scores B1M0A0 (Missing “1 –“ in the denominator so M0)
$\dfrac{\theta\times 2\theta}{1+\left(1-\frac{(3\theta)^2}{2}\right)}$	scores B1M0A0 (Has “1 +“ in the denominator so M0)
$\dfrac{\theta\times \theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \ldots$	scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)
$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \dfrac{2\theta^2}{\frac{9\theta^2}{2}} = \dfrac{2}{9}$	scores B1M1A0 (Missing brackets recovered)
$\dfrac{\theta\times 2\theta}{1-\left(1-\left(\frac{3\theta}{2}\right)^2\right)}$	Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)

$\dfrac{\theta \times 2\theta}{\frac{9\theta^2}{2}} = \dfrac{2}{18}$	B1M1A0 (The B1 is awarded for the numerator but can be implied by the denominator. The M1 is implied)
$\dfrac{\theta \times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \dfrac{4}{9}$	B1M1A1 (The correct value implies correct recovery of missing brackets.)

Note that other approaches are possible using identities.
In such cases we will allow <u>correct</u> work leading to an expression that if terms in θ^3 and higher can be ignored will lead to $\frac{4}{9}$
But to score the M mark they must be using correct identities and correct approximations but condone bracketing errors as in the main scheme.

Examples:

$$\frac{\theta \tan 2\theta}{1-\cos 3\theta} = \frac{\theta \times \frac{\sin 2\theta}{\cos 2\theta}}{1-\cos 3\theta} = \frac{\theta \times \dfrac{2\theta}{1-\frac{(2\theta)^2}{2}}}{1-\left(1-\frac{(3\theta)^2}{2}\right)} = \frac{2\theta^2}{1-2\theta^2} \times \frac{2}{9\theta^2} = \frac{4\theta^2}{9\theta^2 - 18\theta^4}$$

$$= \frac{4\theta^2}{9\theta^2} = \frac{4}{9}$$

Scores B1M1A1

Similarly: $\dfrac{\theta \tan 2\theta}{1-\cos 3\theta} = \dfrac{\theta \times \sin 2\theta}{\cos 2\theta(1-\cos 3\theta)} = \dfrac{\theta \times 2\theta}{\left(1-\frac{(2\theta)^2}{2}\right)\left(1-\left(1-\frac{(3\theta)^2}{2}\right)\right)} = \dfrac{2\theta^2}{1-2\theta^2} \times \dfrac{2}{9\theta^2}$ etc.

$$\frac{4\theta^2}{9\theta^2 - 18\theta^4} = \frac{4\theta^2}{9\theta^2} = \frac{4}{9}$$

Scores B1M1A1

$$\frac{\theta \tan 2\theta}{1-\cos 3\theta} = \frac{\theta \times \sin 2\theta}{\cos 2\theta(1-\cos 3\theta)} = \frac{\theta \times 2\theta}{\left(1-\frac{(2\theta)^2}{2}\right)\left(1-\left(1-\frac{(3\theta)^2}{2}\right)\right)} = \frac{2\theta^2}{1-2\theta^2} \times \frac{2}{9\theta^2}$$

$$= \frac{4\theta^2}{9\theta^2 - 18\theta^4} = \frac{4}{9-18\theta^2} = \frac{4}{9}$$

Scores B1M1A0
(They cannot just assume the term in θ^2 is 0 unless they provide a convincing limiting argument e.g. $\lim\limits_{\theta \to 0} \dfrac{4}{9-18\theta^2} = \dfrac{4}{9}$ or equivalent)

Question ~~18~~ 5 continued

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \approx \frac{2\theta}{1-\theta^2}$$

$$\cos 3\theta = \cos\theta\left(1-2\sin^2\theta\right) - 2\cos\theta\sin^2\theta$$

$$= \cos\theta\left(1-4\sin^2\theta\right)$$

$$\approx \left(1-\frac{\theta^2}{2}\right)\left(1-4\theta^2\right)$$

$$= 1-\frac{9}{2}\theta^2+2\theta^4$$

So $$\frac{\theta\tan 2\theta}{1-\cos 3\theta} = \frac{\frac{2\theta^2}{1-\theta^2}}{1-\left(1-\frac{9}{2}\theta^2+2\theta^4\right)} = \frac{\frac{2\theta^2}{1-\theta^2}}{\frac{9}{2}\theta^2-2\theta^4}$$

$$= \frac{\frac{4}{1-\theta^2}}{9-4\theta^2} = \frac{4}{\left(9-4\theta^2\right)\left(1-\theta^2\right)}$$

$$= \frac{4}{9-13\theta^2+4\theta^4} \approx \frac{4}{9-13\theta^2}$$

Scores B1M1A0

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta} = \frac{\theta\times\frac{2\tan\theta}{1-\tan^2\theta}}{1-\left(4\cos^3\theta-3\cos\theta\right)} = \frac{\theta\times\frac{2\theta}{1-\theta^2}}{1-\left(4\left(1-\frac{\theta^2}{2}\right)^3-3\left(1-\frac{\theta^2}{2}\right)\right)}$$

Scores B1M1A0

Note that attempts to use expansions in higher powers of θ should be sent to review.

Question	Scheme	Marks	AOs
6(a)(i)	$\left(\mathrm{f}'(x)=\right)8x\mathrm{e}^{4x^2-1}$ or e.g. $\dfrac{8x\mathrm{e}^{4x^2}}{\mathrm{e}}$ oe	B1	1.1b
(ii)	$\left(\mathrm{g}'(x)=\right)\dfrac{8}{x}$ or e.g. $8x^{-1}$ oe	B1	1.2
		(2)	
(b)	$8x\mathrm{e}^{4x^2-1}=\dfrac{8}{x}\Rightarrow \mathrm{e}^{4x^2-1}=\dfrac{1}{x^2}\Rightarrow 4x^2-1=\ln\dfrac{1}{x^2}$	M1	1.1b
	$4x^2-1=\ln\dfrac{1}{x^2}\Rightarrow 4x^2-1=-2\ln x$ $\Rightarrow 4x^2+2\ln x-1=0*$	A1*	2.1
		(2)	
(c)(i)	$x_1=0.6\Rightarrow x_2=\sqrt{\dfrac{1-2\ln 0.6}{4}}$	M1	1.1b
	$(x_2=)0.7109$	A1	1.1b
(ii)	$(\alpha=)0.6706$	B1 (A1 In ePEN)	1.1b
		(3)	
		(7 marks)	

(a)(i)

B1: Correct derivative in any form. "$\mathrm{f}'(x)=$" is not required. Apply isw if necessary.

(ii)

B1: Correct derivative in any form. "$\mathrm{g}'(x)=$" is not required. Apply isw if necessary.

(b)

M1: Eliminates e by setting their $\mathrm{f}'(x)=$ their $\mathrm{g}'(x)$ where $\mathrm{f}'(x)=Ax\mathrm{e}^{4x^2-1}$ oe and $\mathrm{g}'(x)=\dfrac{B}{x}$ oe with $A\times B>0$ and proceeds via $\mathrm{e}^{4x^2-1}=\dfrac{\dots}{x^2}$ or equivalent work (see below) to obtain $4x^2-1=\ln\dfrac{\dots}{x^2}$ oe e.g. $\ln x+4x^2-1=\ln\dfrac{1}{x}$

Allow if they use α for x.

Note that there are various alternatives for this mark but the derivatives must be of the form defined above and the processing must be correct with coefficient/sign slips only.

<u>Examples of equivalent work:</u>

$$8x\mathrm{e}^{4x^2-1}=\frac{8}{x}\Rightarrow x^2\mathrm{e}^{4x^2-1}=1\Rightarrow \ln x^2+\ln\mathrm{e}^{4x^2-1}=0\Rightarrow \ln\mathrm{e}^{4x^2-1}=-\ln x^2\Rightarrow 4x^2-1=-2\ln x$$

$$\frac{8x\mathrm{e}^{4x^2}}{\mathrm{e}}=\frac{8}{x}\Rightarrow\frac{1}{\mathrm{e}}\mathrm{e}^{4x^2}=\frac{1}{x^2}\Rightarrow \mathrm{e}^{4x^2}=\frac{\mathrm{e}}{x^2}\Rightarrow \ln\mathrm{e}^{4x^2}=\ln\frac{\mathrm{e}}{x^2}\Rightarrow 4x^2=\ln\frac{\mathrm{e}}{x^2}=1-2\ln x$$

A1*: Obtains the printed answer with sufficient working and no errors.
Sufficient work would require the "e" eliminated before the given answer.
Must follow correct derivatives in part (a).
Condone $4x^2+2\ln|x|-1=0$ and condone $4\alpha^2+2\ln\alpha-1=0$ or $4\alpha^2+2\ln|\alpha|-1=0$

Note that if both derivatives in (a) **<u>are correct</u>** we will allow fully correct work using the equation in (b) to work backwards to verify that $pf'(x) = qg'(x)$ for M1 then obtains $f'(x) = g'(x)$ with a minimal conclusion for A1

If either derivative in (a) is incorrect or missing, candidates who work backwards score no marks in (b).

(c)(i)/(ii)

M1: Attempts to use the iterative formula with $x_1 = 0.6$

Award this mark for e.g. $(x_2 =)\sqrt{\frac{1-2\ln 0.6}{4}}$ or may be implied by awrt 0.71 provided no incorrect working is seen.

Candidates sometimes find x_3 (or possibly subsequent terms) rather than x_2 in which case the M1 can be implied. (See table below for first few iterations)

A1: $(x_2 =)$ awrt 0.7109

Sight of $(x_2 =)$ awrt 0.7109 scores M1A1

B1(A1 on ePEN): (α =) 0.6706 (4dp)

Must be this value and **not** awrt 0.6706

For reference:

x_1	0.6
x_2	0.7109239143
x_3	0.6485329086
x_4	0.6830236199
x_5	0.6637868021
x_6	0.6744606223
. . .	. . .
α	0.6706416243

Question	Scheme	Marks	AOs
7(a)	$\left(\overrightarrow{AB}=\right)3\mathbf{i}+9\mathbf{j}+3\mathbf{k}$	B1	1.1b
		(1)	

Notes for (a)

B1: Correct vector. Allow $3\mathbf{i}+9\mathbf{j}+3\mathbf{k}$ or $\begin{pmatrix}3\\9\\3\end{pmatrix}$ but **not** $\begin{pmatrix}3\mathbf{i}\\9\mathbf{j}\\3\mathbf{k}\end{pmatrix}$ and **not** $(3, 9, 3)$

Condone $\begin{matrix}3\\9\\3\end{matrix}$ for $\begin{pmatrix}3\\9\\3\end{pmatrix}$

Do **not** apply isw here but award for e.g. $3\mathbf{i}+9\mathbf{j}+3\mathbf{k}=\begin{pmatrix}3\mathbf{i}\\9\mathbf{j}\\3\mathbf{k}\end{pmatrix}$

E.g. if they obtain $\overrightarrow{AB}=3\mathbf{i}+9\mathbf{j}+3\mathbf{k}$ and then say $\overrightarrow{AB}=\mathbf{i}+3\mathbf{j}+\mathbf{k}$ then award B0

If part (a) is **not attempted** and the correct $\overrightarrow{AB}$ is seen in part (b) the B1 can be awarded there.

General Guidance for part (b):

As with most vector questions we will see a variety of approaches (correct and incorrect).

In general, the marks are awarded as follows:

- **M1 for a correct complete strategy to find at least one position for *P* (May be implied by at least 2 correct components)**
- **A1 for one correct position for *P***
- **dM1 for a correct complete strategy to find both positions for *P* (May be implied by at least 2 correct components for both positions)**
- **A1 both correct positions for *P* and no others**

Various examples are shown below.
Other methods will be seen but the above marking principles should be applied.
You can condone slips in their algebra/processing as long as the intention is clear.
The examples given below give the detail to look for depending on the approach.
If you see a response and you are not sure if it deserves credit use Review.

Note that adding vectors when they should be subtracting will generally score M0 but use review if necessary.

(b)	Examples: $\overrightarrow{OP}=\overrightarrow{OA}+2\overrightarrow{AB}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}+2(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ or $\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{AB}=5\mathbf{i}+6\mathbf{j}+8\mathbf{k}+(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ or $\overrightarrow{OP}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AB}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}+\frac{2}{3}(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ or $\overrightarrow{OP}=\overrightarrow{OB}+\frac{1}{3}\overrightarrow{BA}=5\mathbf{i}+6\mathbf{j}+8\mathbf{k}-\frac{1}{3}(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$	M1	3.1a
	$8\mathbf{i}+15\mathbf{j}+11\mathbf{k}$ or $4\mathbf{i}+3\mathbf{j}+7\mathbf{k}$	A1	1.1b
	Examples: $\overrightarrow{OP}=\overrightarrow{OA}+2\overrightarrow{AB}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}+2(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ or $\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{AB}=5\mathbf{i}+6\mathbf{j}+8\mathbf{k}+(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ **and** $\overrightarrow{OP}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AB}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}+\frac{2}{3}(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$ or $\overrightarrow{OP}=\overrightarrow{OB}+\frac{1}{3}\overrightarrow{BA}=5\mathbf{i}+6\mathbf{j}+8\mathbf{k}-\frac{1}{3}(3\mathbf{i}+9\mathbf{j}+3\mathbf{k})=...$	dM1	3.1a
	$8\mathbf{i}+15\mathbf{j}+11\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}+7\mathbf{k}$	A1	2.2a
		(4)	
			(5 marks)

Notes for (b)

Note that sight of at least one correct position for *P* implies M1A1

M1: Attempts at least one correct strategy for finding *P*

A1: One correct position vector or allow coordinates for this mark e.g. (8, 15, 11) or (4, 3, 7) or $x = \ldots, y = \ldots, z = \ldots$

If given as a vector, allow e.g. $8\mathbf{i}+15\mathbf{j}+11\mathbf{k}, \begin{pmatrix}8\\15\\11\end{pmatrix}$ but not $\begin{pmatrix}8\mathbf{i}\\15\mathbf{j}\\11\mathbf{k}\end{pmatrix}$

dM1: Attempts two correct strategies for finding *P*

A1: Both correct position **vectors**

Must both be vectors so e.g. $8\mathbf{i}+15\mathbf{j}+11\mathbf{k}, \begin{pmatrix}8\\15\\11\end{pmatrix}$ and $4\mathbf{i}+3\mathbf{j}+7\mathbf{k}, \begin{pmatrix}4\\3\\7\end{pmatrix}$ but not e.g. $\begin{pmatrix}8\mathbf{i}\\15\mathbf{j}\\11\mathbf{k}\end{pmatrix}$

Condone e.g. $\begin{matrix}8\\15\\11\end{matrix}$ for $\begin{pmatrix}8\\15\\11\end{pmatrix}$

Alternative 1 using vector equation of *l*:

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = \begin{pmatrix}2\\-3\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\9\\3\end{pmatrix} \quad \left(\text{or e.g.} \begin{pmatrix}2\\-3\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\1\end{pmatrix}\right)$$

$$\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right| \Rightarrow \left|\begin{pmatrix}3\lambda\\9\lambda\\3\lambda\end{pmatrix}\right| = 2\left|\begin{pmatrix}3\lambda+2-5\\9\lambda-3-6\\3\lambda+5-8\end{pmatrix}\right| \Rightarrow 9\lambda^2 + 81\lambda^2 + 9\lambda^2 = 4\left[(3\lambda-3)^2 + (9\lambda-9)^2 + (3\lambda-3)^2\right]$$

$$\Rightarrow \lambda = 2, \frac{2}{3} \Rightarrow \mathbf{r} = \begin{pmatrix}2\\-3\\5\end{pmatrix} + 2\begin{pmatrix}3\\9\\3\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}2\\-3\\5\end{pmatrix} + \frac{2}{3}\begin{pmatrix}3\\9\\3\end{pmatrix}$$

M1: Forms the vector equation of line *l* using their $\overrightarrow{AB}$ from part (a) or by starting again, forms the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ then uses $\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right|$ and Pythagoras to produce a quadratic equation in "λ" which they then solve to find "λ" and use correctly to find at least one position for *P*.
Note if they use **i** + 3**j** + **k** for the direction, they should get $\lambda = 6, 2$
If all other work is correct, condone not squaring the "2" when applying Pythagoras

A1: See main scheme

dM1: As the first M and finds both positions for *P*

A1: See main scheme

Alternative 2 using *P* as $\begin{pmatrix}x\\y\\z\end{pmatrix}$ and that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are parallel:

$$\overrightarrow{OP} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \overrightarrow{AP} = \begin{pmatrix}x-2\\y+3\\z-5\end{pmatrix}, \ \overrightarrow{BP} = \begin{pmatrix}x-5\\y-6\\z-8\end{pmatrix}$$

$$\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right| \Rightarrow \left|\begin{pmatrix}x-2\\y+3\\z-5\end{pmatrix}\right| = 2\left|\begin{pmatrix}x-5\\y-6\\z-8\end{pmatrix}\right| \Rightarrow \begin{matrix}(x-2)^2 = 4(x-5)^2\\(y+3)^2 = 4(y-6)^2\\(z-5)^2 = 4(z-8)^2\end{matrix}$$

$$\begin{matrix}(x-2)^2 = 4(x-5)^2 \Rightarrow x = 4, 8\\(y+3)^2 = 4(y-6)^2 \Rightarrow y = 3, 15\\(z-5)^2 = 4(z-8)^2 \Rightarrow z = 7, 11\end{matrix} \Rightarrow \begin{pmatrix}4\\3\\7\end{pmatrix}, \begin{pmatrix}8\\15\\11\end{pmatrix}$$

M1: Sets *P* as a general point, forms $\overrightarrow{AP}$ and $\overrightarrow{BP}$ (either way round) then uses $\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right|$ then squares components and equates to produce quadratic equations in *x* and *y* and *z* which they then solve to find at least one position for *P*. It is not just for finding values which are not then used to form a point (or vector).
If all other work is correct, condone not squaring the "2" when squaring.

A1: See main scheme

dM1: As the first M and finds both positions for *P*.

A1: See main scheme

Note that if the modulus is not used, this method can lead to one correct position for P e.g.

$$\overrightarrow{AP} = 2\overrightarrow{BP} \Rightarrow \begin{pmatrix} x-2 \\ y+3 \\ z-5 \end{pmatrix} = 2\begin{pmatrix} x-5 \\ y-6 \\ z-8 \end{pmatrix} \Rightarrow \begin{matrix} x=8 \\ y=15 \\ z=11 \end{matrix} \text{ and scores M1 A1}$$

But it is possible to find the other position without squaring e.g.

$$\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right| \Rightarrow \begin{pmatrix} x-2 \\ y+3 \\ z-5 \end{pmatrix} = 2\begin{pmatrix} 5-x \\ 6-y \\ 8-z \end{pmatrix} \Rightarrow \begin{matrix} x=4 \\ y=3 \\ z=7 \end{matrix} \text{ and scores dM1 then A1 as main scheme.}$$

This requires at least 2 correct equations for x, y or z for the dM1

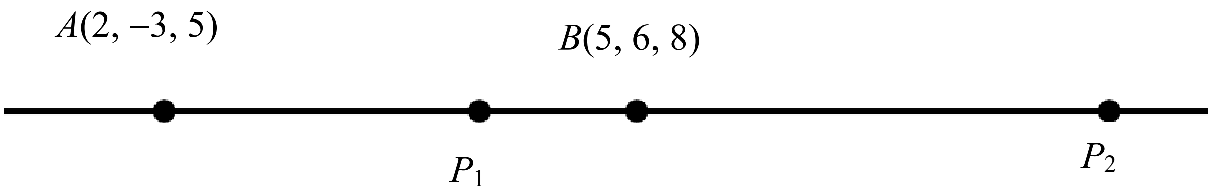
e.g. Alternative 3 using P as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and eliminating 2 of the variables:

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \overrightarrow{AP} = \begin{pmatrix} x-2 \\ y+3 \\ z-5 \end{pmatrix},\ \overrightarrow{BP} = \begin{pmatrix} x-5 \\ y-6 \\ z-8 \end{pmatrix}$$

$$\overrightarrow{AP} = \begin{pmatrix} x-2 \\ y+3 \\ z-5 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{AP} = \begin{pmatrix} x-2 \\ 3x-6 \\ x-2 \end{pmatrix},\ \overrightarrow{BP} = \begin{pmatrix} x-5 \\ y-6 \\ z-8 \end{pmatrix} = \mu\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{BP} = \begin{pmatrix} x-5 \\ 3x-15 \\ x-5 \end{pmatrix}$$

$$\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{BP}\right| \Rightarrow (x-2)^2 + (3x-6)^2 + (x-2)^2 = 4\left[(x-5)^2 + (3x-15)^2 + (x-5)^2\right] \Rightarrow x = 4,\ 8$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}\left(\text{or } \overrightarrow{OB} + \overrightarrow{BP}\right) = \begin{pmatrix} x \\ 3x-9 \\ x+3 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} \text{ or } \begin{pmatrix} 8 \\ 15 \\ 11 \end{pmatrix}$$



Question	Scheme	Marks	AOs
8(a) Way 1	$\left(\frac{1}{\text{cosec}\,\theta-1}+\frac{1}{\text{cosec}\,\theta+1}\equiv\right)\frac{\text{cosec}\,\theta+1+\text{cosec}\,\theta-1}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}$	B1	1.1b
	$\equiv\frac{2\text{cosec}\,\theta}{\text{cosec}^2\theta-1}\equiv\frac{2\text{cosec}\,\theta}{\cot^2\theta}$ or e.g. $\equiv\frac{2\sin\theta}{1-\sin^2\theta}=\frac{2\sin\theta}{\cos^2\theta}$	M1	1.1b
	$\frac{2\text{cosec}\,\theta}{\cot^2\theta}\equiv\frac{2}{\sin\theta}\times\frac{\sin^2\theta}{\cos^2\theta}\equiv 2\tan\theta\sec\theta*$ or $\frac{2\sin\theta}{\cos^2\theta}\equiv 2\tan\theta\sec\theta*$	A1*	2.1
		(3)	
(a) Way 2	**"Meets in the middle"**		
	$\left(\text{LHS}=\frac{1}{\text{cosec}\,\theta-1}+\frac{1}{\text{cosec}\,\theta+1}\equiv\right)\frac{\text{cosec}\,\theta+1+\text{cosec}\,\theta-1}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}$	B1	1.1b
	$\equiv\frac{2\text{cosec}\,\theta}{\text{cosec}^2\theta-1}\equiv\frac{2\text{cosec}\,\theta}{\cot^2\theta}$	M1	1.1b
	$\text{RHS}=2\tan\theta\sec\theta\equiv\frac{2\sin\theta}{\cos^2\theta}\equiv\frac{2\sin^2\theta}{\sin\theta\cos^2\theta}$ $\equiv\frac{2}{\sin\theta}\times\frac{\sin^2\theta}{\cos^2\theta}\equiv\frac{2\,\text{cosec}\,\theta}{\cot^2\theta}=\text{LHS}$ or e.g. QED or e.g. Proven	A1*	2.1
		(3)	

Part (a) Notes

(a) **Condone a complete proof entirely in x (or another variable) instead of θ**
Condone "=" for "≡"

Note that we are marking this as **B1M1A1** not **M1M1A1**

B1: Adds the fractions to obtain a **correct** single fraction (not fractions over fractions) in any form. Condone missing brackets when they combine their fractions as long as they are recovered to give a correct fraction.

This can be done in a variety of ways but when combined, the fraction must be **correct** e.g.

$$\frac{\text{cosec}\,\theta+1+\text{cosec}\,\theta-1}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}\text{ or }\frac{2\text{cosec}\,\theta}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}\text{ or }\frac{2\text{cosec}\,\theta}{\text{cosec}^2\theta-1}$$

$$\text{or e.g. }\left(\frac{1}{\frac{1}{\sin\theta}-1}+\frac{1}{\frac{1}{\sin\theta}+1}=\frac{\sin\theta}{1-\sin\theta}+\frac{\sin\theta}{1+\sin\theta}\right)=\frac{2\sin\theta}{1-\sin^2\theta}\text{ etc.}$$

M1: **Uses** a **correct** Pythagorean identity anywhere in their attempt e.g. $\text{cosec}^2\theta-1=\cot^2\theta$, $\sin^2\theta+\cos^2\theta=1$ etc. or equivalent

A1*: Correct work with all necessary steps shown leading to the given answer. See scheme for the necessary steps. They need to proceed via sine and cosine to the given answer. There should be **no notational or bracketing errors and no mixed or missing variables**. E.g. we would consider $\cos^2\theta$ written as $\cos\theta^2$ a notational error.
Condone reaching $2\sec\theta\tan\theta*$

Way 2 (Meet in the middle)

B1: See Way 1

M1: See Way 1

A1*: Correct work on the RHS with all necessary steps shown leading to showing the equivalence with the LHS. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.
For this approach there must be a (minimal) conclusion e.g. "= LHS", "QED", "Hence proven" etc.

It is possible to start with the rhs e.g.:

$$2\tan\theta\sec\theta = 2\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta}$$

$$= \frac{2\sin\theta}{\cos^2\theta} = \frac{2\operatorname{cosec}\theta}{\cot^2\theta}$$

$$= \frac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta - 1} = \frac{2\operatorname{cosec}\theta}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$$

$$= \frac{\operatorname{cosec}\theta + 1 + \operatorname{cosec}\theta - 1}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$$

$$= \frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1}$$

B1: Correctly reaches $2\tan\theta\sec\theta = \dfrac{2\operatorname{cosec}\theta}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$

M1: **Uses** a **correct** Pythagorean identity anywhere in their attempt e.g. $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$, $\sin^2\theta + \cos^2\theta = 1$ etc. or equivalent

A1*: Correct work with all necessary steps shown leading to the lhs. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.

8(b)	$2\tan 2x\sec 2x = \cot 2x\sec 2x$	B1	2.2a
	$2\tan 2x\sec 2x - \cot 2x\sec 2x = 0$ $\Rightarrow \sec 2x\left(2\tan 2x - \cot 2x\right) = 0$ $2\tan 2x - \cot 2x = 0 \Rightarrow 2\tan 2x = \cot 2x \Rightarrow \tan^2 2x = \frac{1}{2}$ **or** $2\tan 2x - \cot 2x = 0 \Rightarrow 2\frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2\sin^2 2x = \cos^2 2x$ $\Rightarrow 2\sin^2 2x = 1 - \sin^2 2x \Rightarrow \sin^2 2x = \frac{1}{3}$ **or** $2\left(1 - \cos^2 2x\right) = \cos^2 2x \Rightarrow \cos^2 2x = \frac{2}{3}$ **or** $2\tan 2x = \cot 2x \Rightarrow \frac{4\tan x}{1 - \tan^2 x} = \frac{1 - \tan^2 x}{2\tan x}$ $\Rightarrow \tan^4 x - 10\tan^2 x + 1 = 0$ **or** $2\tan 2x - \cot 2x = 0 \Rightarrow 2\frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2\sin^2 2x = \cos^2 2x$ $2\sin^2 2x = \cos^2 2x \Rightarrow 1 - \cos 4x = \frac{1}{2}\left(\cos 4x + 1\right) \Rightarrow \cos 4x = \frac{1}{3}$	M1	2.1
	$2x = \tan^{-1}\frac{1}{\sqrt{2}} = K \Rightarrow x = \frac{K}{2}$ **or** $2x = \sin^{-1}\frac{1}{\sqrt{3}} = K \Rightarrow x = \frac{K}{2}$ **or** $2x = \cos^{-1}\sqrt{\frac{2}{3}} = K \Rightarrow x = \frac{K}{2}$ **or** $\tan^2 x = 5 \pm 2\sqrt{6} \Rightarrow x = \tan^{-1}\left(\sqrt{5 \pm 2\sqrt{6}}\right)$ **or** $\cos 4x = \frac{1}{3} \Rightarrow 4x = \cos^{-1}\left(\frac{1}{3}\right) = K \Rightarrow x = \frac{1}{4}K$	M1	1.1b
	$x = 17.6°,\ 72.4°$	A1	1.1b
		(4)	
		(7 marks)	

(b) Notes

(b) Note that attempts solve an equation of the form:
$2\tan x\sec x=\cot 2x\sec 2x$ or e.g. $2\tan\theta\sec\theta=\cot 2\theta\sec 2\theta$ or e.g. $2\tan\theta\sec\theta=\cot 2x\sec 2x$
Will generally score no marks in part (b)

Condone the use of θ instead of x here.

B1: Deduces the correct equation using the result from part (a)

M1: Factors out or cancels the $\sec 2x$ to obtain $...\tan 2x\pm...\cot 2x=0$ oe e.g. $...\tan 2x=\pm...\cot 2x$ leading to an equation of the form:
$\tan^2 2x=\alpha$ oe e.g. $\cot^2 2x=\frac{1}{\alpha}$ where $\alpha>0$

Or uses $\tan 2x=\frac{\sin 2x}{\cos 2x}$ and $\cot 2x=\frac{\cos 2x}{\sin 2x}$ and then $\sin^2 2x=\pm 1\pm\cos^2 2x$ or $\cos^2 2x=\pm 1\pm\sin^2 2x$ to obtain an equation of the form $\sin^2 2x=\beta$ or $\cos^2 2x=\gamma$ oe e.g. $\text{cosec}^2 2x=\frac{1}{\beta}$ or $\sec^2 2x=\frac{1}{\gamma}$ where $0<\beta<1$ or $0<\gamma<1$

Or uses $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$ and $\cot 2x=\frac{1-\tan^2 x}{2\tan x}$ to obtain a 3TQ in $\tan^2 x$ (or possibly in $\sec^2 x$)

Or uses $\tan 2x=\frac{\sin 2x}{\cos 2x}$ and $\cot 2x=\frac{\cos 2x}{\sin 2x}$ and then $2\sin^2 2x=\pm 1\pm\cos 4x$ and $2\cos^2 2x=\pm 1\pm\cos 4x$ to obtain an equation of the form $\cos 4x=k,\quad 0<k<1$

M1: Correct order of operations from $\tan^2 2x=\alpha$ or $\sin^2 2x=\beta$ or $\cos^2 2x=\gamma$ or $\cos 4x=k$ or equivalents e.g. $\text{cosec}^2 2x=\frac{1}{\beta}$ where $\alpha>0$ or $0<\beta<1$ or $0<\gamma<1$ or $0<k<1$
leading to at least one value for x e.g. square roots, finds inverse tan/sin/cos/cosec and divides by 2 or inverse cos and divides by 4
or from $\tan^2 x=k,\ k>0$ which follows their equation (may need to check) and then finds $x=\tan^{-1}\sqrt{k}$
You may need to check their value(s) (in degrees or radians) to see if the correct order of operations has been used. May be implied by e.g. $17.6°$ or $17.7°$ provided no incorrect work is seen.

A1: Correct values. Allow awrt $17.6°$ and awrt $72.4°$. The degrees symbol is not required. Ignore any values outside the range (correct or incorrect) but if there are extra angles in range score A0. Answers in radians score A0.

Note that some candidates may convert to $\sin x$ or $\cos x$ and then solve:

E.g.

$$\tan^2 2x=\frac{1}{2}\Rightarrow\frac{\sin^2 2x}{\cos^2 2x}=\frac{1}{2}\Rightarrow\frac{4\sin^2 x\cos^2 x}{\left(2\cos^2 x-1\right)^2}\rightarrow 12\cos^4 x-12\cos^2 x+1=0$$

$$\text{or }\frac{4\sin^2 x\cos^2 x}{\left(1-2\sin^2 x\right)^2}\rightarrow 12\sin^4 x-12\sin^2 x+1=0$$

$$\Rightarrow\cos^2 x/\sin^2 x=\frac{3\pm\sqrt{6}}{6}\Rightarrow\cos x/\sin x=\pm\sqrt{\frac{3\pm\sqrt{6}}{6}}\Rightarrow x=17.6°,\ 72.4°$$

These can be marked in a similar way.

Alternative not using part (a):

$$\frac{1}{\text{cosec}2x-1}+\frac{1}{\text{cosec}2x+1}=\cot 2x\sec 2x$$

$$\Rightarrow \frac{2\text{cosec}2x}{\text{cosec}^2 2x-1}=\cot 2x\sec 2x$$

$$\Rightarrow \frac{2\text{cosec}2x}{\cot^2 2x}=\frac{\cos 2x}{\sin 2x}\times\frac{1}{\cos 2x}=\text{cosec}2x$$

$$\Rightarrow 2\tan^2 2x=1\Rightarrow \tan^2 2x=\frac{1}{2}$$

Score as:

M1: For correct work leading to one of the forms in the main scheme e.g.

$\tan^2 2x=\alpha$ oe e.g. $\cot^2 2x=\frac{1}{\alpha}$ where $\alpha>0$

or

$\sin^2 2x=\beta$ or $\cos^2 2x=\gamma$ oe e.g. $\text{cosec}^2 2x=\frac{1}{\beta}$ or $\sec^2 2x=\frac{1}{\gamma}$

where $0<\beta<1$ or $0<\gamma<1$

B1: Any correct equation e.g. $\tan^2 2x=\frac{1}{2}$, $\cot^2 2x=2$, $\sin^2 2x=\frac{1}{3}$ etc.

Then **M1A1** as main scheme

Question	Scheme	Marks	AOs
9(a) Way 1	$H = \pm ax^2 \pm bx \pm c$ $x = 0, H = 2 \Rightarrow c = 2$ **and** either $x = 20, H = 0.8 \Rightarrow 0.8 = 400a + 20b + 2$ **or** $H = ax^2 + bx + c \Rightarrow \frac{dH}{dx} = 2ax + b$ $x = 9, \frac{dH}{dx} = 0 \Rightarrow 18a + b = 0$	M1	3.3
	$H = \pm ax^2 \pm bx \pm c$ $x = 0, H = 2 \Rightarrow c = 2$ **and** $x = 20, H = 0.8 \Rightarrow 0.8 = 20^2 a + 20b + 2$ **and** $H = ax^2 + bx + c \Rightarrow \frac{dH}{dx} = 2ax + b$ $x = 9, \frac{dH}{dx} = 0 \Rightarrow 18a + b = 0$	dM1	3.1b
	$0.8 = 400a + 20b + 2,\ 18a + b = 0 \Rightarrow a = ..., b = ...$	ddM1	1.1b
	$H = -0.03x^2 + 0.54x + 2$	A1	2.2a
		(4)	

(a) Way 1 Notes

Condone use of *y* for *H* for the <u>method</u> marks.

A model of the form $H = x^2 + ax + b$ or $H = -x^2 + ax + b$ will score no marks.

Note that it is possible to identify (by symmetry) that the points (–2, 0.8) and (18, 2) also lie on the parabola so you may see valid use of these points.

M1: Uses the equation $H = \pm ax^2 \pm bx \pm c$ to model the path and uses $x = 0$ and $H = 2$ **correctly placed** to establish the value of the constant term and uses $x = 20$ and $H = 0.8$ **or** $x = 9, \frac{dH}{dx} = 0$ to give an equation in 'a' and 'b' with $\frac{dH}{dx}$ of the form $...\alpha x + \beta$

An alternative is to recognise that the maximum occurs when $x = -\frac{b}{2a} = 9$ or equivalent

e.g. maximum when $x = 9 \Rightarrow H = a(x-9)^2 + ... = ax^2 - 18ax + ... \Rightarrow b = -18a$

Award for $\pm\frac{b}{2a} = 9$ or equivalent.

They may also use e.g. (–2, 0.8) or (18, 2) to give an equation in a and b.

dM1: This mark requires:

- uses the equation $H = \pm ax^2 \pm bx \pm c$ to model the path and uses $x = 0$ and $H = 2$ **correctly placed** to establish the value of the constant term
- uses $x = 20$ and $H = 0.8$ **correctly placed and** $x = 9, \frac{dH}{dx} = 0$ to give 2 equations in 'a' and 'b' with $\frac{dH}{dx}$ of the form $...ax + b$ or as above using $\pm\frac{b}{2a} = 9$

They may also use e.g. (–2, 0.8) or (18, 2) to give an equation in a and b.

ddM1: Solves their 2 equations in "a" and "b" to find their 'a' and 'b'.
This may be done on a calculator. You do **not** need to check their method for solving.

A1: Correct **equation**. Must be $H = f(x)$.

(a)	$x = 9$ at max $\Rightarrow H = A \pm B(x-9)^2$		
Way 2	**and** either $x = 0, H = 2 \Rightarrow 2 = A + 81B$ **or** $x = 20, H = 0.8 \Rightarrow 0.8 = A + 121B$	M1	3.3
	$x = 9$ at max $\Rightarrow H = A + B(x-9)^2$ **and** $x = 0, H = 2 \Rightarrow 2 = A + 81B$ **and** $x = 20, H = 0.8 \Rightarrow 0.8 = A + 121B$	dM1	3.1b
	$2 = A + 81B,\ 0.8 = A + 121B \Rightarrow A = 4.43,\ B = -0.03$	ddM1	1.1b
	$H = 4.43 - 0.03(x-9)^2$	A1	2.2a
		(4)	

(a) Way 2 Notes

Condone use of *y* for *H* for the method marks.
A model of the form $H = A \pm (x-9)^2$ will score no marks.

M1: Uses the equation $H = A \pm B(x-9)^2$ or $H = A \pm B(9-x)^2$ to model the path and uses one of the 'end points' **correctly placed** to give an equation in 'A' and 'B'
They may also use e.g. (–2, 0.8) or (18, 2) to give an equation in A and B.

dM1: Uses the equation $H = A + B(x-9)^2$ or $H = A + B(9-x)^2$ to model the path and uses both 'end points' **correctly placed** to give 2 equations in 'A' and 'B'
They may also use e.g. (–2, 0.8) or (18, 2) to give an equation in A and B.

ddM1: Solves their 2 equations in "A" and "B" to find their 'A' and 'B'.
This may be done on a calculator. You do **not** need to check their method for solving.

A1: Correct **equation**. Must be $H = f(x)$.

Note that using $H = A + B(x-9)^2$ followed by the incorrect assumption that $A = 2$ is unlikely to score any marks as they will subsequently not be able to produce 2 equations in "A" and "B"

Possible alternative 3:

$$H = A\left((x-9)^2 - 81\right) + B$$
$$x = 0, H = 2 \Rightarrow B = 2$$
$$x = 20, H = 0.8 \Rightarrow 0.8 = 40A + B$$
$$B = 2 \Rightarrow A = -0.03$$
$$H = 2 - 0.03\left((x-9)^2 - 81\right)$$

M1: Uses the equation $H = A\left((x-9)^2 - 81\right) + B$ to model the path and uses $H = 2$ when $x = 0$ **correctly placed** to find "B"

dM1: Uses the equation $H = A\left((x-9)^2 - 81\right) + B$ to model the path and uses $H = 0.8$ when $x = 20$ **correctly placed.** May also use e.g. (–2, 0.8) or (18, 2)

ddM1: Substitutes their value for "B" to find a value for "A"

A1: Correct **equation**. Must be $H = f(x)$.

(b)	Examples must focus on why the model may not be appropriate or give situations where the model would break down e.g.: • *H* is unlikely to be a quadratic function in *x* • The path is unlikely to be parabolic • Wind may affect the path of the ball • Wind may affect the distance the ball travels • Air resistance has not been considered • The ball is unlikely to travel in a vertical plane (as it may spin) • The ball is not a particle so has dimensions/size • The ground is unlikely to be horizontal • There may be trees (or other hazards) that would affect the path of the ball • The shape of the ball may affect the motion Condone statements (where the link to the model is not completely made) such as • The ball will spin • Ground is not flat • The ball is not a particle Do **not** accept statements that refer to the situation outside the range of the throw e.g. • The model is not valid for all values of *x* • *H* will become negative Do **not** accept statements that do not refer to the given model or single word vague answers e.g. • The distances may have been measured incorrectly • The ball is not modelled as a particle • "Friction", "Spin", "Force", "air resistance" • It does not take into account the weight of the ball • It depends how good the thrower is • You cannot throw the ball the same way every time	B1	3.5b
		(1)	
(c)	$x = 16 \Rightarrow H = -0.03(16)^2 + 0.54(16) + 2 = ...$	M1	3.4
	$H = 2.96$ So Chandra would not be able to catch the ball	A1	3.2a
		(2)	
			(7 marks)

Notes for (b) and (c)

(b)

B1: Gives a suitable limitation – see scheme
If more than one limitation is given and one is acceptable then award this mark as long as none of the other statements are contradictory (they may be incorrect/inappropriate)

(c)

M1: Substitutes $x = 16$ into their equation modelling the path **to obtain a value** for H. This may be seen explicitly as above or may be implied by their value (you may need to check). **Must have a quadratic function in *x*.**

A1: Depends on

- A correct equation
- $H = 2.96$
- Correct conclusion that she cannot catch the ball or equivalent

A minimum for M1A1 could be e.g. $x = 16 \Rightarrow H = 2.96$ "so no"

(c) Alternative:

$$\text{e.g. } 2.5 = 4.43 - 0.03(x-9)^2 \Rightarrow x = 9 + \frac{\sqrt{579}}{3} = 17.02...$$

So Chandra would not be able to catch the ball

M1: Substitutes $H = 2.5$ into their **quadratic** equation modelling the path **to obtain a value** for x. This may be seen explicitly as above or may be implied by their value (you may need to check). **Must have a quadratic function in *x*.**

A1: Depends on

- A correct equation
- $x =$ awrt 17
- Correct conclusion that she cannot catch the ball or equivalent**.**

A minimum for M1A1 could be e.g. $H = 2.5 \Rightarrow x = 17$ "so no"

Question	Scheme	Marks	AOs
10(a)	$x=4, y=2 \Rightarrow t=-1$	B1	2.2a
	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=-3t^2\times\frac{1}{2(t+3)}$	M1	1.1b
	$\frac{dy}{dx}=-3(-1)^2\times\frac{1}{2(-1+3)}=-\frac{3}{4}$	M1	1.1b
	$\Rightarrow y-2=-\frac{3}{4}(x-4)$ or $\Rightarrow y=-\frac{3}{4}x+c \rightarrow 2=-\frac{3}{4}\times 4+c \Rightarrow c...$	ddM1	2.1
	$y-2=-\frac{3}{4}(x-4)\Rightarrow 4y-8=-3x+12$ or $c=5\Rightarrow y=-\frac{3}{4}x+5$ $\Rightarrow 3x+4y=20*$	A1*	1.1b
		(5)	
(b)	Maximum height is 9m	B1	3.4
		(1)	
		(6 marks)	

Notes

(a) **<u>If parametric differentiation is not used in part (a) (e.g. uses Cartesian form) then only the B mark is available but see alternative below.</u>**

B1: Uses the given Cartesian coordinates to deduce the correct value for t.
If more than one value for t e.g. $t=-5$ is given and $t=-1$ is not "selected" score B0 but if **just** $t=-1$ is used subsequently allow recovery and score B1

M1: Attempts to use $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$ or equivalent with their differentiated equations.
There must be an attempt to differentiate both parameters, however poor, and divide or multiply correctly so using $\frac{dy}{dx}=\frac{y}{x}$ scores M0. Both parameters must be "changed".
Condone confusion with the variables e.g. referring to $\frac{dy}{dt}$ as $\frac{dy}{dx}$ if the intention is clear.
This may be implied by e.g. $\frac{dy}{dt}=-3t^2, \frac{dx}{dt}=2(t+3), t=-1, \frac{dy}{dt}=-3, \frac{dx}{dt}=4\Rightarrow\frac{dy}{dx}=-\frac{3}{4}$

M1: Uses their numerical value of t (not 4) in their $\frac{dy}{dx}$ to obtain a value.
Condone attempts with different values of t e.g. $t=-1$ and $t=-5$

ddM1: Applies a correct straight line method with their value of $\frac{dy}{dx}$ which has come from an attempt to use parametric differentiation with their value of t (not 4) and with $x = 4$ and $y = 2$ correctly placed. An attempt at the equation of the normal is M0.
If using $y = mx + c$ they must reach as far as $c = \ldots$
Depends on both previous M marks.

A1*: Correct equation as printed with no errors but condone $4y+3x=20*$
Allow equivalents e.g. $20=4y+3x*$ or $3x+4y=20*$
This is a printed answer so there must be at least one intermediate step as shown in the main scheme.

Alternative for (a) using parametric differentiation but avoids the need for a value for *t*:

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=-3t^2\times\frac{1}{2(t+3)}$$

$$-3t^2\times\frac{1}{2(t+3)}=-3(1-y)^{\frac{2}{3}}\times\frac{1}{2\sqrt{x}}=-3(1-2)^{\frac{2}{3}}\times\frac{1}{2\sqrt{4}}=-\frac{3}{4}$$

or

$$-3t^2\times\frac{1}{2(t+3)}=-3\left(\sqrt{x}-3\right)^2\times\frac{1}{2\sqrt{x}}=-3(2-3)^2\times\frac{1}{2\sqrt{4}}=-\frac{3}{4}$$

or

$$-3t^2\times\frac{1}{2(t+3)}=-3(1-y)^{\frac{2}{3}}\times\frac{1}{2\left((1-y)^{\frac{1}{3}}+3\right)}=-3(1-2)^2\times\frac{1}{2\times 2}=-\frac{3}{4}$$

$$\Rightarrow y-2=-\frac{3}{4}(x-4)\Rightarrow 3x+4y=20*$$

B1: **Either** a correct expression for $\frac{dy}{dx}$ in terms of *x* and/or *y* following a correct $\frac{dy}{dx}$ in terms of *t* **or** for $t=-1$ seen anywhere.

M1: Attempts to use $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$ with their differentiated equations.
There must be an attempt to differentiate both parameters, however poor, and divide or multiply correctly so using $\frac{dy}{dx}=\frac{y}{x}$ scores M0

Condone confusion with the variables e.g. referring to $\frac{dy}{dt}$ as $\frac{dy}{dx}$ if the intention is clear.

This may be implied by e.g. $\frac{dy}{dt}=-3t^2$, $\frac{dx}{dt}=2(t+3)$, $t=-1$, $\frac{dy}{dt}=-3$, $\frac{dx}{dt}=4\Rightarrow\frac{dy}{dx}=-\frac{3}{4}$

M1: Attempts to express their $\frac{dy}{dx}$ which is in terms of *t*, in terms of *x* and/or *y* **and** uses $x = 4$ and $y = 2$ correctly placed in an attempt to find the gradient of the tangent.

ddM1: Applies a correct straight line method with their value of $\frac{dy}{dx}$ which has come from an attempt to use parametric differentiation with their gradient and with $x = 4$ and $y = 2$ correctly placed.
If using $y = mx + c$ they must reach as far as $c = \ldots$
Depends on both previous M marks.

A1*: Correct equation as printed with no errors.
This is a printed answer so there must be at least one intermediate step as shown in the main scheme.

(b)

B1: 9m or equivalent **including correct units**. Accept e.g. 9 metres, 900cm etc.

Question	Scheme	Marks	AOs
11	$\int 8x^2 e^{-3x}\,dx = -\frac{8x^2}{3}e^{-3x} + \int \frac{16x}{3}e^{-3x}\,dx$	M1 A1	2.1 1.1b
	$= -\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} + \int \frac{16}{9}e^{-3x}\,dx$	**d**M1	1.1b
	$\left[-\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} - \frac{16}{27}e^{-3x}\right]_0^1$ $= -\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-0 - 0 - \frac{16}{27}\right)$	M1	2.1
	$= \frac{16}{27} - \frac{136}{27}e^{-3}$	A1	1.1b
		(5)	
		(5 marks)	

Notes

Mark positively in this question and do not penalise poor notation such as a missing "d*x*" or spurious integral signs, "+ *c*" etc. as long as the intention is clear.

M1: Obtains $\pm\alpha x^2 e^{-3x} \pm \beta \int x\,e^{-3x}\,dx$

(you do not need to be concerned about how they arrive at this)

A1: Correct expression **simplified or unsimplified**. E.g. allow $-\frac{8x^2}{3}e^{-3x} - \int -\frac{16x}{3}e^{-3x}\,dx$

Note that we condone the "8" missing for this mark so allow e.g. $-\frac{x^2}{3}e^{-3x} - \int -\frac{2x}{3}e^{-3x}\,dx$

Note that notation may be poor here but the intention clear e.g. if they obtain $-\frac{8x^2}{3}e^{-3x} + \left[\frac{16x}{3}e^{-3x}\right]$ and then attempt to integrate $\frac{16x}{3}e^{-3x}$ both marks can be implied.

dM1: Attempts parts again on $\pm\beta\int x\,e^{-3x}\,dx$ to obtain $\pm Axe^{-3x} \pm B\int e^{-3x}\,dx$

This may be seen in isolation and does not need to be seen as part of the complete integration. **Depends on the first method mark.**

Watch for the DI method (with or without the 8):

	D	I
+	$8x^2$	e^{-3x}
−	$16x$	$-\frac{1}{3}e^{-3x}$
+	16	$\frac{1}{9}e^{-3x}$
−	0	$-\frac{1}{27}e^{-3x}$

Giving the correct integration e.g. $\int 8x^2 e^{-3x}\,dx = -\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} - \frac{16}{27}e^{-3x}$

In such cases score M1dM1 for obtaining $\pm px^2 e^{-3x} \pm qxe^{-3x} \pm re^{-3x},\ p,q,r \neq 0$ and then A1 for the **correct** first 2 terms, with or without the factor of 8.

Note that for this approach M1A1dM0 is not possible.

M1: Substitutes the limits 1 and 0 into an expression of the form $\pm\alpha x^2 e^{-3x} \pm \beta x e^{-3x} \pm \gamma e^{-3x}$, $\alpha,\beta,\gamma \neq 0$ and subtracts the right way round.
Must see evidence of the use of **both** limits and subtraction and use of $e^0 = 1$.
Note that some candidates apply the limits as they go e.g. to the $\left[-\frac{8x^2}{3}e^{-3x}\right]$ which is acceptable but you will need to check carefully that overall they are satisfying the conditions above.
Condone not realising that the first 2 terms evaluate to 0 when substituting $x = 0$ e.g.
condone $-\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-\frac{8}{3} - \frac{16}{9} - \frac{16}{27}\right)$ as we have evidence of $e^0 = 1$
Note that e.g. $-\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-\frac{16}{27}e^0\right) = -\frac{136}{27}e^{-3} - \frac{16}{27}$ scores M0 as it suggests that $e^0 = -1$ not $+1$.

A1: Correct answer of $\frac{16}{27} - \frac{136}{27}e^{-3}$ but allow equivalent exact fractions and condone $\frac{16}{27} - \frac{136}{27e^3}$. Isw once the correct answer is seen.

Candidates who consistently misread $8x^2e^{-3x}$ as $8x^2e^{3x}$:

$$\int 8x^2e^{3x}\,dx = \frac{8x^2}{3}e^{3x} - \int \frac{16x}{3}e^{3x}\,dx$$
$$= \frac{8x^2}{3}e^{3x} - \frac{16x}{9}e^{3x} + \int \frac{16}{9}e^{3x}\,dx$$
$$\left[\frac{8x^2}{3}e^{3x} - \frac{16x}{9}e^{3x} + \frac{16}{27}e^{3x}\right]_0^1$$
$$= \frac{8}{3}e^3 - \frac{16}{9}e^3 + \frac{16}{27}e^3 - \left(\frac{16}{27}\right) = \frac{40}{27}e^3 - \frac{16}{27}$$

Scores a maximum of M1A0dM1M1A0
The main scheme can be applied similarly e.g.

M1: Attempts parts to obtain $\alpha x^2 e^{3x} - \beta \int x\, e^{3x}\,dx$, $\alpha, \beta > 0$

A0: Not available

dM1: Attempts parts again on $\beta \int x\, e^{3x}\,dx$ to obtain $Cxe^{3x} - D\int e^{3x}\,dx$, $C, D > 0$

M1: Substitutes the limits 1 and 0 into an expression of the form $\pm\lambda x^2 e^{3x} \pm \mu x e^{3x} \pm \gamma e^{3x}$, $\lambda,\mu,\gamma \neq 0$ and subtracts the right way round.
Must see evidence of the use of **both** limits and subtraction and use of $e^0 = 1$.

A0: Not available

But note, do **not** allow mixing of $3x$'s and $-3x$'s. If there are a mixture, apply the main scheme.

Question	Scheme	Marks	AOs
12(a)	$\frac{1}{V(25-V)} = \frac{P}{V} + \frac{Q}{25-V}$ e.g. $1 = P(25-V) + QV$ $V = 0$ or $V = 25$ leading to $P = \ldots$ or $Q = \ldots$	M1	1.1b
	$\frac{1}{V(25-V)} = \frac{1}{25V} + \frac{1}{25(25-V)}$	A1	1.1b
		(2)	

Notes

(a)

M1: Sets $\frac{1}{V(25-V)} = \frac{P}{V} + \frac{Q}{25-V}$ and uses a correct method to identify the value of at least one constant.

Do **not** condone incorrect work e.g. $\frac{1}{V(25-V)} = \frac{P}{V} + \frac{Q}{25-V} \Rightarrow 1 = PV + Q(25-V)$ etc.

this scores M0

A1: Correct partial fractions in any form e.g.

$\frac{1}{25V} + \frac{1}{25(25-V)}, \frac{1}{25V} + \frac{1}{625-25V}, \frac{1/25}{V} + \frac{1/25}{(25-V)}, \frac{1}{25V} - \frac{1}{25(V-25)}, \frac{1}{25}\left(\frac{1}{V} + \frac{1}{25-V}\right)$ etc.

Note that this mark is not just for the correct constants, it is for the correctly written fractions either seen in part (a) **or used in part (b).** Allow 0.04 for $\frac{1}{25}$.

Correct partial fractions only scores both marks.

If the correct fractions are obtained following incorrect work score M0A0 but allow full recovery in the rest of the question.

(b) Way 1	$\int \frac{1}{V(25-V)} \mathrm{d}V = \int \frac{1}{25V} + \frac{1}{25(25-V)} \mathrm{d}V = \frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)$	M1	3.1a
	$\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V) = \frac{1}{10}t(+c)$	A1ft	1.1b
	$t=0, V=20 \Rightarrow \frac{1}{25}\ln 20 - \frac{1}{25}\ln(25-20) = c \left(\Rightarrow c = \frac{1}{25}\ln 4\right)$	M1	3.4
	$V = 24 \Rightarrow t = \frac{2}{5}\ln 24 - \frac{2}{5}\ln(25-24) - \frac{2}{5}\ln 4$	**d**M1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
		(5)	
	Alternative for the final 3 marks:		
	$\left[\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\frac{1}{10}t\right]_0^T \Rightarrow \frac{1}{25}\ln 24 - \frac{1}{25}\ln 4 = \frac{1}{10}T$	M1	3.4
	$T = \frac{10}{25}\ln 24 - \frac{10}{25}\ln 4 = \ldots$	**d**M1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
(c)	$\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V) = \frac{1}{10}t + \frac{1}{25}\ln 4$ $\ln V - \ln(25-V) = 2.5t + \ln 4$ $\ln\frac{V}{4(25-V)} = 2.5t \Rightarrow \frac{V}{4(25-V)} = \mathrm{e}^{2.5t}$	M1	2.1
	$\Rightarrow V = 4\mathrm{e}^{2.5t}(25-V) \Rightarrow V + 4V\mathrm{e}^{2.5t} = 100\mathrm{e}^{2.5t} \Rightarrow V = \ldots$	M1	2.1
	$\Rightarrow V = \frac{100\mathrm{e}^{2.5t}}{1+4\mathrm{e}^{2.5t}} = \frac{100}{\mathrm{e}^{-2.5t}+4}$	A1	1.1b
		(3)	
(d)	25 (microlitres)	B1	2.2a
	Since e.g. As $t \to \infty$, $\mathrm{e}^{"-2.5"t} \to 0$	B1	2.4
		(2)	
			(12 marks)

Notes

(b) **Mark (b) and (c) together**

M1: Realises that $\int \frac{\ldots}{V(25-V)}(\mathrm{d}V)$ is required and reaches the form $p\ln\alpha V \pm q\ln\beta(25-V)$ (or e.g. $p\ln\alpha V \pm q\ln\beta(V-25)$) or equivalent for this integration e.g. $p\ln 25V - q\ln(625-25V)$ with p and q non-zero.

But note that $\int \frac{\ldots}{V(25-V)}\mathrm{d}V = \ln V(25-V)$ does not score this mark unless we see an attempt to integrate the partial fractions first.

Condone missing brackets e.g. around the $V-25$ for this mark

Note that the rhs may be incorrect or missing for this mark.

A1ft: Fully correct **equation** following through their P and Q. The "+ c" is **not** required here. You may need to check carefully when awarding this mark as there will be various alternative correct (or correct ft) forms e.g. these are correct (for correct PF's):

$\frac{1}{25}\ln 25V - \frac{1}{25}\ln(625-25V) = \frac{1}{10}t(+c)$, $\ln V - \ln(25-V) = 2.5t(+c)$,

$\frac{2}{5}\ln 25V - \frac{2}{5}\ln(25-V) = t(+c)$, $\frac{2}{5}\ln 5V - \frac{2}{5}\ln(125-5V) = t(+c)$

In general look for an equation of the form $P\ln\alpha V - Q\ln\beta(25-V) = \frac{1}{10}t(+c)$ or a multiple of this equation. Do **not** condone missing brackets unless they are implied by later work e.g. $\ln 25 - V$ for $\ln(25-V)$

Allow ( ) or | | around the arguments of the ln's and condone "log" for ln.

M1: States or uses $t = 0$ and $V = 20$ consistently leading to a constant of integration which may be simplified or unsimplified. May be implied by their constant so may need to be checked.
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: States or uses $V = 24$ and proceeds to find a value for t (even if $t < 0$). You do not need to check the processing provided they reach a value for t. **Depends on the previous method mark** and depends on an attempt to integrate both sides however poor.
May be implied by their value for t so may need to be checked.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

Alternative for final 3 marks:

M1: $\left[\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\frac{1}{10}t\right]_0^T \Rightarrow \frac{1}{25}\ln 24 - \frac{1}{25}\ln 4 = \frac{1}{10}T$
Applies the limits 20 and 24 to lhs and 0 to "T" or e.g. "t" on rhs
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: $T = \frac{10}{25}\ln 24 - \frac{10}{25}\ln 4 = ...$ Solves to find "T". You do not need to check the processing provided they reach a value for t/T. **Depends on the previous method mark** and depends on having made some attempt to integrate both sides however poor.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

Note

$$\int \frac{1}{25V} + \frac{1}{25(25-V)}\,dV = \int \frac{1}{25V} - \frac{1}{25V-625}\,dV = \frac{1}{25}\ln V - \frac{1}{25}\ln(25V-625) = \frac{1}{10}t(+c)$$

is also correct integration and scores M1A1

The subsequent marks are also available as described above and could lead to the correct answer if the limits and constant of integration are dealt with correctly.
Use review for any examples like these if you are unsure but generally apply the MS as above.

(c) The marks in (c) depend on having integrated their partial fractions to obtain an equation of the form $\pm...\ln...V \pm ...\ln...(25-V) = \pm kt \pm c,\quad k, c \neq 0$ and … are non-zero constants or equivalent if they have already attempted to eliminate the ln's in (b)

e.g. $\frac{...V}{...(25-V)} = e^{\pm kt \pm c}$ oe

M1: Uses fully correct log work, having obtained a constant of integration, to eliminate all the ln's including from e.g. $e^{\ln 4}$. We condone sign or coefficient slips only.

M1: Proceeds from an **equation of the form** $\frac{...V}{...(25-V)} = ...e^{...t}$ oe using correct algebra to

$V = \ldots$ e.g. $\frac{...V}{...(25-V)} = ...e^{...t} \Rightarrow ...V = ...(25-V)...e^{...t} \Rightarrow (...\pm...)V = ...e^{...t} \Rightarrow V = ...$

Condone sign/coefficient slips only.

A1: Correct expression not just values for the constants.

(d)

B1: Correct value of 25 seen
Allow e.g. < 25 or „ 25
Condone "> 25" but the following mark is then not available

B1: Depends on a correct final equation in any form in (c) e.g. $V = \frac{100e^{2.5t}}{1+4e^{2.5t}}$ oe **and one of**:

- Considers the behaviour as $t \to \infty$ e.g. states that as $t \to \infty$, $e^{-2.5t} \to 0$ (condone "= 0") oe
- $V < 25$ as ln(25 – V) is not possible when $V \ldots 25$
- Verifies the 25 using a value of t, $t \ldots 9$

Using the differential equation:

B1: Correct value of 25 seen
Allow e.g. "< 25" or „ 25
Condone "> 25" but the following mark is then not available

B1: E.g. when $V = 25$, $\frac{dV}{dt} = 0$ **or** $\frac{dV}{dt} < 0$ if $V > 25$

Question	Scheme	Marks	AOs
13(a)	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ **or** $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	3.1a
	$b = 1.01$ **or** $a = 6.46$	A1	1.1b
	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ **and** $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	2.1
	$b = 1.013$ **and** $a = 6.457$	A1	1.1b
		(4)	
(b)(i)	e.g. The world population **in billions** in 2004	B1ft	3.2a
(ii)	*b* = 1.013 represents the scale factor of the yearly increase in the world population	B1ft	3.2a
		(2)	
(c)	$P = 6.457...(1.013...)^{26}$ or e.g. $\log P = 0.81 + 26 \times 0.0054 \Rightarrow P = ...$	M1	3.4
	awrt 9 billion	A1	2.2b
		(2)	
(d)	Not reliable since the data used for the model covered the years 2004 – 2007 and it would not be sensible to assume that the model still holds in 2030	B1	3.2b
		(1)	
			(9 marks)

Notes

(a) Must be using base 10 in (a). Ignore any units associated with *a* and *b* in part (a).

M1: Correct strategy to get a numerical expression or value for *a* **or** *b* e.g. $a = 10^{0.81}$ **or** $b = 10^{0.0054}$. This may be implied by *a* = awrt 6.46 or *b* = awrt 1.01 if no incorrect work is seen.

A1: Correct value for *a* or *b*. Allow 3 sf for **this mark** so allow *a* = awrt 6.46 or *b* = awrt 1.01.
May be seen embedded in their formula.

M1: Correct strategy to get a numerical expression or value for *a* **and** *b* e.g. $a = 10^{0.81}$ **and** $b = 10^{0.0054}$. This may be implied by *a* = awrt 6.46 and *b* = awrt 1.01 if no incorrect work is seen.

A1: Correct values. This requires *a* = awrt 6.457 and *b* = awrt 1.013 for this mark.
May be seen embedded in their formula.
Isw once correct answers are seen.

Special case: Constants the wrong way round:

$a = 1.013$ **and** $b = 6.457$ with or without working scores M1A1M1A0 unless the equation is formed correctly in which case the final A mark can be recovered.

Note that having found the value of *a*, it is possible to find *b* by substituting e.g. *t* = 1 as follows:

$$a = 10^{0.81} = 6.457 \quad t = 1 \Rightarrow P = ab \Rightarrow b = \frac{P}{a}$$

$$t = 1 \Rightarrow \log_{10} P = 0.0054 + 0.81 = 0.8154 \Rightarrow P = 10^{0.8154} \Rightarrow b = \frac{P}{a} = \frac{10^{0.8154}}{6.457} = 1.013$$

Note that a misread of 0.0054 as 0.054 is quite common and may score 1110 as it does not simplify the question.

(b)(i) Follow through their *a*.

B1ft: Correct interpretation for *a* but must reference "billions".
Allow equivalent alternatives e.g.

- The original/initial population **in billions**
- The population in 2004 was "6.46" **billion**

(b)(ii) Follow through their *b*.

B1ft: Correct interpretation for *b* but must reference "each year" or e.g. "yearly" oe
Allow equivalent alternatives e.g.

- The proportional increase/change in each year.
- The population will rise by "1.3%" each year. Must follow their value for *b*.
- The rate/factor at which the population is rising/increasing/changing per annum.
- "1.013" is the multiplier representing the year on year increase.

Do **not** accept

- The amount it is rising
- How much it is rising
- The rate the population increases
- The percentage increase each year
- The rate of increase in billions annually

If they are not labelled (b)(i) and (b)(ii) mark in the order given but accept any way round as long as clearly labelled " *a* is........... " and "*b* is ...."

(c)

M1: Substitutes $t = 25$ or 26 or 27 into their model to find a value for P
Must be using **their** a and b correctly in $P = ab^t$
May be implied by sight of "9" or 9 billion if no incorrect working is seen.

A1: Correct value including units (allow awrt 9 billion) **from a correct model but condone incorrect/premature rounding or truncating in an otherwise correct model that leads to the correct value of awrt 9 billion.**
Allow e.g. awrt 9 000 000 000 or e.g. awrt 9×10^9
Just awrt 9 without the "billions" is A0

(d)

B1: The response must refer to the fact that the answer is unreliable together with a reference to the fact that the data used for the model is a long way from 2030
Examples:

- Not good as 2030 is a long way from 2004 – 2007
- Unreliable as based on old data
- Questionable as it has been extrapolated over a long time
- Not reliable due to how far out we have extrapolated
- By the time 2030 arrives it will be unreliable

But not e.g.

- Unreliable, extrapolation
- Not good as outside the range
- Not good as the population rises 101.3% each year
- Disease may happen
- Reliable as based on old data

Question	Scheme	Marks	AOs
14(a)(i)	Centre is (3, −7)	B1	1.1b
(ii)	$(x-3)^2+(y+7)^2=49+9-33 \Rightarrow r^2=...(25)$	M1	1.1b
	$r=5$	A1	1.1b
		(3)	
(b)	Distance between centres = $\sqrt{(3+6)^2+(-7+8)^2}=\sqrt{82}$	M1 A1ft	3.1a 1.1b
	$"\sqrt{82}"-"5"$ **or** $"\sqrt{82}"+"5"$	dM1	3.1a
	$\sqrt{82}-5$ **and** $\sqrt{82}+5$	A1	2.2a
	$\{k:\sqrt{82}-5<k\}\cap\{k:k<\sqrt{82}+5\}$ or e.g. $\{k:\sqrt{82}-5<k<\sqrt{82}+5\}$	A1	2.5
		(5)	
		(8 marks)	

Notes

(a)(i)

B1: Correct centre. Allow as a coordinate pair or written separately e.g. $x = 3, y = -7$ or as a column vector $\begin{pmatrix}3\\-7\end{pmatrix}$

Condone missing brackets e.g. 3, −7 but do **not** allow coordinates the wrong way round.

(ii)

M1: Uses a correct strategy to find the radius or radius^2

Requires an attempt at: $(x\pm3)^2+(y\pm7)^2-3^2-7^2\pm33=0 \Rightarrow (x\pm3)^2+(y\pm7)^2=\alpha,\ \alpha>0$

Award for $(x\pm3)^2+(y\pm7)^2-a^2-b^2\pm33=0 \Rightarrow (x\pm3)^2+(y\pm7)^2=\alpha,\ \alpha>0$ with at least one of $a = 3$ or $b = 7$ (or 9 or 49)

You may see an attempt at $"f^2+g^2-c"$ or $\sqrt{"f^2+g^2-c"}$ e.g. $"3^2+7^2\pm33"$ or $\sqrt{"3^2+7^2\pm33"}$

A1: Correct radius of 5. Do **not** allow ±5 or $\sqrt{25}$.

May be scored following $(x\pm3)^2+(y\pm7)^2=25$

Correct answers only in (a) scores B1M1A1

(b)

M1: Uses Pythagoras correctly on their centre from part (a) and the given centre to find the distance between the centres.

Look for $\sqrt{(-6-(\text{their } x))^2+(-8-(\text{their } y))^2}$ or e.g. $\sqrt{((\text{their } x)-(-6))^2+((\text{their } y)-(-8))^2}$

but condone one sign slip with their coordinates if the intention is clear.

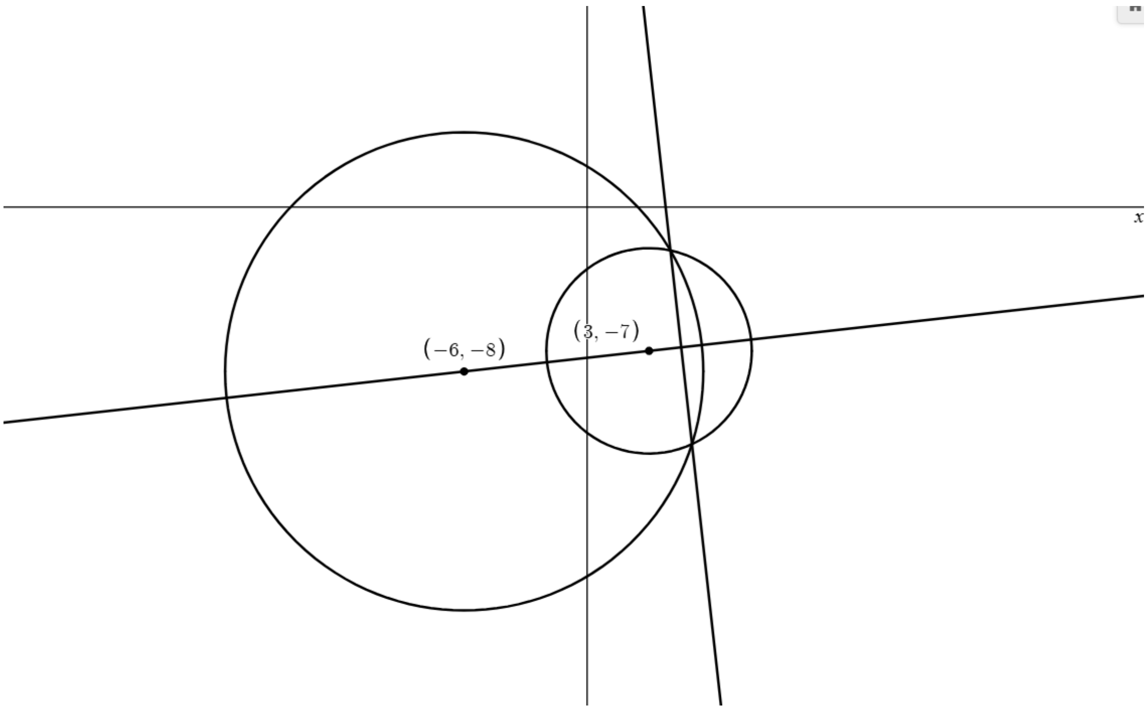
A1ft: Correct distance or follow through their centre from part (a).

This may be implied by their value. Condone the use of decimals so allow 3sf accuracy e.g. awrt 9.06 for $\sqrt{82}$ or you may need to check their value following an incorrect centre in (a)(i). Not e.g. $\pm\sqrt{82}$ unless the positive root is subsequently used.

dM1: Correct strategy for one of the limits. E.g. adds or subtracts their 5 to their distance between centres.

A1:	Correct limits. There is no follow through but allow decimals to 3sf e.g. awrt 4.06 and awrt 14.1
A1:	Correct answer with **exact values** using set notation. Allow as shown in the main scheme but also allow equivalent set notation e.g. $\{k : k \in \mathrm{R}, \sqrt{82}-5<k<\sqrt{82}+5\}$, $\{k :\ \sqrt{82}-5<k<\sqrt{82}+5\}$, $k \in (\sqrt{82}-5,\ \sqrt{82}+5)$ and allow "|" for ":" and allow the "k:" or "$k \in$" to be missing e.g. $\{\sqrt{82}-5<k<\sqrt{82}+5\}$ and $(\sqrt{82}-5,\ \sqrt{82}+5)$ are both acceptable. But $\{k : k<\sqrt{82}+5\} \cup \{k : k>\sqrt{82}-5\}$ or $\{k :\ k<\sqrt{82}-5,\ k<\sqrt{82}+5\}$ score A0 Do **not** allow solutions not in set notation such as $\sqrt{82}-5<k<\sqrt{82}+5$ **Correct answers with no working should be sent to review.**

Scenario for part (b) for reference:



Algebraic approach for part (b):

$$x^2+y^2-6x+14y+33=x^2+12x+36+y^2+16y+64-k^2$$

$$\Rightarrow 18x+2y+67-k^2=0 \Rightarrow y=\frac{k^2-67}{2}-9x$$

$$(x-3)^2+(y+7)^2=25 \Rightarrow x^2-6x+9+\left(\frac{k^2-67}{2}-9x+7\right)^2=25$$

$$\Rightarrow 82x^2+471x-9k^2x+\frac{k^4-106k^2+2745}{4}=0$$

When circles touch $b^2 - 4ac = 0$

$$\Rightarrow \left(471-9k^2\right)^2-4\times 82\left(\frac{k^4-106k^2+2745}{4}\right)=0$$

$$\Rightarrow k^4-214k^2+3249=0$$

$$\Rightarrow \left(k^2-10k-57\right)\left(k^2+10k-57\right)=0$$

$$\Rightarrow k=5+\sqrt{82},\ 5-\sqrt{82},\ -5+\sqrt{82},\ -5-\sqrt{82}$$

$$k=\underline{5+\sqrt{82},\ -5+\sqrt{82}}$$

We will mark this as follows:

M1: This requires a valid strategy that:

- solves the 2 circle equations simultaneously to find y in terms of x and k, or x in terms of y and k
- substitutes for y or x into one of the circle equations to obtain an equation in x and k only, or y and k only,
- attempts $b^2-4ac=0$ or e.g. $b^2-4ac>0$ or equivalent to obtain an equation in k only. You do not need to look at the details of their algebra.

A1: Correct simplified 3TQ in k^2

dM1: Solves their 3TQ in k^2 by any correct method including a calculator to find k.

A1: Both correct values for k (exact or decimals as in the main scheme) (they may have extras which can be ignored)

A1: As main scheme (exact and in set notation)

Note that work such as

$$(x+6)^2+(y+8)^2=k^2 \Rightarrow x+6+y+8=k$$

is not a valid strategy as it greatly simplifies the problem and would generally score no marks.

Implicit differentiation approach for part (b):

$$x^2+y^2-6x+14y+33=0 \Rightarrow 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}-6+14\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6-2x}{2y+14}$$

$$(x+6)^2+(y+8)^2=k^2 \Rightarrow 2(x+6)+2(y+8)\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x+6}{y+8}$$

$$\frac{3-x}{y+7}=-\frac{x+6}{y+8} \Rightarrow (3-x)(y+8)=-(x+6)(y+7) \Rightarrow x=9y+66$$

$$(9y+66)^2+y^2-6(9y+66)+14y+33=0 \Rightarrow 82y^2+1148y+3993=0$$

(or $82x^2-492x-1287=0$)

$$y=\frac{-574\pm5\sqrt{82}}{82} \Rightarrow x=\frac{246\pm45\sqrt{82}}{82}$$

$$\left(\frac{246+45\sqrt{82}}{82},\frac{-574+5\sqrt{82}}{82}\right) \to \left(\frac{246+45\sqrt{82}}{82}+6\right)^2+\left(\frac{-574+5\sqrt{82}}{82}+8\right)^2=k^2$$

$$\Rightarrow k^2=107+10\sqrt{82} \Rightarrow k=5+\sqrt{82}$$

Then the same for $\left(\frac{246-45\sqrt{82}}{82},\frac{-574-5\sqrt{82}}{82}\right) \to k=-5+\sqrt{82}$

We will mark this as follows:

M1: This requires a valid strategy that:

- differentiates the equations of both circles implicitly and equates the derivatives to obtain an equation connecting *y* and *x*. (Note that the equation connecting *y* and *x* is the common equation through the centres which can also be found from using the coordinates of the centres)
- substitutes for *x* or *y* into the equation for C_1 to obtain an equation in one variable

A1: Correct 3TQ in *y* or *x*

dM1: This requires:

- solves their 3TQ in *y* or *x* by any correct means including a calculator and finds at least one point of intersection
- substitutes this point into C_2 and proceeds to a value for *k*

A1: Correct values for *k* (exact or decimals as in the main scheme)

A1: As main scheme (exact and in set notation)

Question	Scheme	Marks	AOs
15(a)	$3(x+y)^2\left(1+\dfrac{dy}{dx}\right)=6x-3\dfrac{dy}{dx}$	M1 A1 A1	3.1a 1.1b 1.1b
	$\left(3(x+y)^2+3\right)\dfrac{dy}{dx}=6x-3(x+y)^2 \Rightarrow \dfrac{dy}{dx}=...$	M1	2.1
	$\dfrac{dy}{dx}=\dfrac{6x-3(x+y)^2}{3(x+y)^2+3}$ $\left(\text{oe e.g. } \dfrac{2x-(x+y)^2}{(x+y)^2+1}\right)$	A1	1.1b
		(5)	
	Alternative – expands $(x+y)^3$ before differentiating $(x+y)^3=x^3+3x^2y+3xy^2+y^3$		
	$\Rightarrow 3x^2+3x^2\dfrac{dy}{dx}+6xy+6xy\dfrac{dy}{dx}+3y^2+3y^2\dfrac{dy}{dx}=6x-3\dfrac{dy}{dx}$	M1 A1 A1	3.1a 1.1b 1.1b
	$\left(3x^2+6xy+3y^2+3\right)\dfrac{dy}{dx}=6x-3x^2-6xy-3y^2 \Rightarrow \dfrac{dy}{dx}=...$	M1	2.1
	$\left(3x^2+6xy+3y^2+3\right)\dfrac{dy}{dx}=6x-3x^2-6xy-3y^2$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{6x-3x^2-6xy-3y^2}{3x^2+6xy+3y^2+3}$ $\left(\text{oe e.g. } \dfrac{2x-x^2-2xy-y^2}{x^2+2xy+y^2+1}\right)$	A1	1.1b

(a) Notes

(a) Some candidates have a spurious "$\dfrac{dy}{dx}=$" appearing as their intention to differentiate e.g.

$$\left(\frac{dy}{dx}=\right)3(x+y)^2\left(1+\frac{dy}{dx}\right)=6x-3\frac{dy}{dx}$$

This can be condoned for the first 3 marks in both versions.

Allow equivalent notation for the $\dfrac{dy}{dx}$ e.g. y'

M1: Award this mark for one of:

- $(x+y)^3 \to k(x+y)^2\left(\lambda+\dfrac{dy}{dx}\right)$ where λ is 1, x or 0 but condone missing brackets e.g. $3(x+y)^2 1+\dfrac{dy}{dx}$
- $3x^2-3y-2 \to ...x+...\dfrac{dy}{dx}$ but condone $3x^2-3y-2 \to ...x+...\dfrac{dy}{dx}-2$

A1: **Either** $3(x+y)^2\left(1+\dfrac{dy}{dx}\right)$ **or** $6x-3\dfrac{dy}{dx}$ oe

May be implied if e.g. they collect terms to one side initially.
Do not condone missing brackets unless they are implied by subsequent work.

A1: $3(x+y)^2\left(1+\dfrac{dy}{dx}\right)$ **and** $6x-3\dfrac{dy}{dx}$ (seen separately or equated)

If they collect terms to one side initially then the signs must be correct.

M1:	A valid attempt to make $\frac{dy}{dx}$ the subject with exactly 2 **different** terms in $\frac{dy}{dx}$, one coming from the differentiation of $(x+y)^3$ and the other coming from the differentiation of "$-3y$" Note that here, 2 **different** terms means terms such as $3\frac{dy}{dx}$ and $3(x+y)^2\frac{dy}{dx}$ and not e.g. $3\frac{dy}{dx}$ and $-8\frac{dy}{dx}$ Look for $(...\pm...)\frac{dy}{dx}=...\Rightarrow\frac{dy}{dx}=...$ which may be implied by their working. Condone slips provided the intention is clear. For those candidates who had a spurious $\frac{dy}{dx}=...$ at the start, they may incorporate this in their rearrangement in which case they will have 3 terms in $\frac{dy}{dx}$ and so score M0. If they ignore it, then this mark is available for the condition as described above. Note that from $3(x+y)^2\left(1+\frac{dy}{dx}\right)=6x-3\frac{dy}{dx}$, candidates may expand the brackets before rearranging, in which case they would need 4 **different** $\frac{dy}{dx}$ terms coming from the appropriate places. Note that the different $\frac{dy}{dx}$ terms do not have to be correct as long as the above conditions are satisfied.
A1:	Fully correct expression for $\frac{dy}{dx}$. Allow any equivalent correct forms. Apply isw as soon as a correct expression is seen.
(a) alternative by expanding:	
M1:	Award this mark for one of: • $3x^2-3y-2\rightarrow...x+...\frac{dy}{dx}$ but condone $3x^2-3y-2\rightarrow...x+...\frac{dy}{dx}-2$ • Expanding $(x+y)^3$ to obtain either an x^2y term or an xy^2 term and then uses the product rule to obtain $...x^2y\rightarrow...x^2\frac{dy}{dx}+...xy$ or $...xy^2\rightarrow...xy\frac{dy}{dx}+...y^2$
A1:	**Either** $3x^2+3x^2\frac{dy}{dx}+6xy+6xy\frac{dy}{dx}+3y^2+3y^2\frac{dy}{dx}$ **or** $6x-3\frac{dy}{dx}$. May be implied if e.g. they collect terms to one side initially.
A1:	$3x^2+3x^2\frac{dy}{dx}+6xy+6xy\frac{dy}{dx}+3y^2+3y^2\frac{dy}{dx}$ **and** $6x-3\frac{dy}{dx}$ oe. (seen separately or equated) If they collect terms to one side initially then the signs must be correct.
M1:	A valid attempt to make $\frac{dy}{dx}$ the subject with exactly 4 **different** terms in $\frac{dy}{dx}$, 3 coming from the differentiation of $(x+y)^3$ and the other coming from the differentiation of "$-3y$" Note that here, 4 **different** terms means terms such as $x^2\frac{dy}{dx}$ and $6xy\frac{dy}{dx}$ and not e.g.

$3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $-8\frac{\mathrm{d}y}{\mathrm{d}x}$

Look for $(\ldots\pm\ldots\pm\ldots\pm\ldots)\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots$ which may be implied by their working.

Condone slips provided the intention is clear.

For those candidates who had a spurious $\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots$ at the start, they may incorporate this in their rearrangement in which case they will have 5 terms in $\frac{\mathrm{d}y}{\mathrm{d}x}$ and so score M0.

If they ignore it, then this mark is available for the condition as described above.

Note that the different $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms do not have to be correct as long as the above conditions are satisfied. E.g. if they have an incorrect term such as $6x\frac{\mathrm{d}y}{\mathrm{d}x}$, this mark is still available.

A1: Fully correct expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Allow any equivalent correct forms.

Condone e.g. $3x2y$ for $6xy$.

Apply isw as soon as a correct expression is seen.

Alternative making *y* the subject in (a):

$$(x+y)^3=3x^2-3y-2$$

$$x+y=\left(3x^2-3y-2\right)^{\frac{1}{3}}\Rightarrow y=\left(3x^2-3y-2\right)^{\frac{1}{3}}-x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}\left(3x^2-3y-2\right)^{-\frac{2}{3}}\left(6x-3\frac{\mathrm{d}y}{\mathrm{d}x}\right)-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\left(3x^2-3y-2\right)^{-\frac{2}{3}}\right)=2x\left(3x^2-3y-2\right)^{-\frac{2}{3}}-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x\left(3x^2-3y-2\right)^{-\frac{2}{3}}-1}{1+\left(3x^2-3y-2\right)^{-\frac{2}{3}}}$$

Score as follows:

M1: Cube roots both sides and makes $x + y$ or y the subject then award for

- $\left(3x^2-3y-2\right)^{\frac{1}{3}}\to\ldots\left(3x^2-3y-2\right)^{-\frac{2}{3}}$ or
- $3x^2-3y-2\to\ldots x+\ldots\frac{\mathrm{d}y}{\mathrm{d}x}$ but condone $3x^2-3y-2\to\ldots x+\ldots\frac{\mathrm{d}y}{\mathrm{d}x}-2$

A1: For the $\frac{1}{3}\left(3x^2-3y-2\right)^{-\frac{2}{3}}$ **or** $6x-3\frac{\mathrm{d}y}{\mathrm{d}x}$

A1: Fully correct

M1: A valid attempt to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject with exactly 2 **different** terms in $\frac{\mathrm{d}y}{\mathrm{d}x}$

A1: Correct expression

Using partial derivatives in (a):

$$(x+y)^3 = 3x^2 - 3y - 2 \rightarrow \mathrm{f}(x,y) = (x+y)^3 - 3x^2 + 3y + 2$$

$$\frac{\partial \mathrm{f}}{\partial x} = 3(x+y)^2 - 6x \qquad \frac{\partial \mathrm{f}}{\partial y} = 3(x+y)^2 + 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial \mathrm{f}}{\partial x} \div \frac{\partial \mathrm{f}}{\partial y} = \frac{6x - 3(x+y)^2}{3(x+y)^2 + 3}$$

or

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 3x^2 - 3y - 2$$

$$\mathrm{f}(x,y) = x^3 + 3x^2y + 3xy^2 + y^3 - 3x^2 + 3y + 2$$

$$\frac{\partial \mathrm{f}}{\partial x} = 3x^2 + 6xy + 3y^2 - 6x \qquad \frac{\partial \mathrm{f}}{\partial y} = 3x^2 + 6xy + 3y^2 + 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial \mathrm{f}}{\partial x} \div \frac{\partial \mathrm{f}}{\partial y} = \frac{-3x^2 - 6xy - 3y^2 + 6x}{3x^2 + 6xy + 3y^2 + 3}$$

Score as follows:

M1: Correct structure for either partial derivative:

Doesn't expand: $\frac{\partial \mathrm{f}}{\partial x} = ...(x+y)^2 + ...x$ or $\frac{\partial \mathrm{f}}{\partial y} = ...(x+y)^2 + ...$

or

Expands: $\frac{\partial \mathrm{f}}{\partial x} = ...x^2 + ...xy + ...y^2 + ...x$ or $\frac{\partial \mathrm{f}}{\partial y} = ...x^2 + ...xy + ...y^2 + ...$

Where "…" are non-zero constants

A1: Correct $\frac{\partial \mathrm{f}}{\partial x}$ **or** correct $\frac{\partial \mathrm{f}}{\partial y}$

A1: Correct $\frac{\partial \mathrm{f}}{\partial x}$ **and** correct $\frac{\partial \mathrm{f}}{\partial y}$

M1: Attempts $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial \mathrm{f}}{\partial x} \div \frac{\partial \mathrm{f}}{\partial y}$

A1: Correct expression

(b)	$\frac{dy}{dx}=\frac{6(1)-3(0+1)^2}{3(0+1)^2+3}=\frac{1}{2}$ or e.g. $\frac{dy}{dx}=\frac{6(1)-3(1)^2-6(1)(0)-3(0)^2}{3(1)^2+6(1)(0)+3(0)^2+3}=\frac{1}{2}$ $\Rightarrow y-0=-2(x-1)$ **or** $\Rightarrow y=-2x+c\Rightarrow 0=-2+c\Rightarrow c=...$	M1	2.1
	$y=-2x+2^*$	A1*	1.1b
		(2)	

(b) Notes

(b) **Note that the gradient of $\frac{1}{2}$ could have been deduced from the given equation so you will need to check their solution carefully.**

M1: Substitutes $x = 1$ and $y = 0$ into their $\frac{dy}{dx}$ to obtain the tangent gradient **and** then uses the negative reciprocal and $x = 1$ and $y = 0$ in a correct straight line method to obtain the normal equation with $x = 1$ and $y = 0$ correctly placed.
Note that when finding the normal gradient, they may find the negative reciprocal of their expression from part (a) and then substitute $x = 1$ and $y = 0$ which is fine.
If using $y = mx + c$ they must proceed as far as finding a value for c.
If no substitution of $x = 1$ and $y = 0$ into their $\frac{dy}{dx}$ is seen you will need to check their value. If they **just** state a value for $\frac{dy}{dx}$ then it must follow their $\frac{dy}{dx}$ with $x = 1$and $y = 0$

A1*: Correct equation with no errors <u>**following a correct**</u> $\frac{dy}{dx}$ from part (a) (unless they start again which is unlikely)
Be aware that some incorrect expressions for $\frac{dy}{dx}$ from part (a) may fortuitously give $\frac{dy}{dx}=\frac{1}{2}$ and would generally score A0
In general A1* must follow the final A1 in (a) or correct differentiation in (a)

(c)	$y=-2x+2 \Rightarrow (x-2x+2)^3 = 3x^2-3(-2x+2)-2$ **or** $x=\dfrac{2-y}{2} \Rightarrow \left(\dfrac{2-y}{2}+y\right)^3 = 3\left(\dfrac{2-y}{2}\right)^2 - 3y-2$	M1	1.1b
	$x^3-3x^2+18x-16=0$ **or** $y^3+60y=0$	A1	1.1b
	$\Rightarrow (x-1)(x^2-2x+16)=0$ (x = 1 is known) or $\Rightarrow y(y^2+60)=0$ (y = 0 is known)	dM1	2.1
	For $x^2-2x+16=0,\ \ b^2-4ac=4-4\times1\times16$ or For $y^2+60=0,\ \ y^2 \neq -60$	ddM1	2.1
	As $b^2-4ac<0$ or as $y^2 \neq -60$ there are no other real roots and so the normal does not meet C again.	A1	2.4
		(5)	

(c) Notes

(c)

M1: Uses the equation from part (a) and substitutes $y=\pm2x\pm2$ or $x=\dfrac{\pm2\pm y}{2}$ to obtain an equation in one variable (usually x) (not necessarily a cubic equation). Allow slips in rearranging to obtain x in terms of y (or y in terms of x) as long as the intention is clear.

A1: Correct cubic equation with terms collected and "= 0" seen or implied.
Note that both $-x^3+3x^2-18x+16=0$ and $-y^3-60y=0$ are correct equations.

To access any of the following marks, candidates must attempt to use either the factor of (x – 1) with their cubic in x or the factor of y in their cubic in y to obtain a quadratic expression in x or y.
Attempts that just use a calculator to solve the cubic equation score no more marks in this part.

dM1: Uses the fact that (x – 1) or y is a factor in an attempt to establish the quadratic factor.
For the cubic in x, it must be of the form $ax^3+bx^2+cx+d=0 \quad a,b,c,d\neq0$
For the cubic in y, it must be of the form $ay^3+by=0 \quad a,b\neq0$
For the cubic in x, the attempt at the quadratic factor using (x – 1) may be via inspection or e.g. long division to obtain a 3 term quadratic expression. There may or may not be a remainder but they must obtain 3 terms.
For the cubic in y, they would need to take out a factor of y (or divide through by y) to obtain a factor of the form $k(y^2+\alpha)$

ddM1: This mark requires:

- a correct cubic equation in x or y
- the correct quadratic factor or a multiple of it e.g. $k\left(x^2-2x+16\right)$ or $k\left(y^2+60\right)$
- an attempt to show that the quadratic factor has no real roots

For the quadratic in *x* this could be:

Attempts discriminant: e.g. $b^2-4ac=4-4\times1\times16$ (may be embedded in the quadratic formula)

Attempts to complete the square: e.g. $x^2-2x+16=(x-1)^2-1+16$

Uses calculus to find the turning point: e.g. $\frac{\mathrm{d}\left(x^2-2x+16\right)}{\mathrm{d}x}=2x-2=0\Rightarrow x=1\Rightarrow y=...$

Attempts to solve: e.g. $x^2-2x+16=0\Rightarrow x=\frac{2\pm\sqrt{4-4\times16}}{2}$ or from a calculator $x=1\pm\sqrt{15}\mathrm{i}$

For the quadratic in *y* this is likely to be:

Attempts to solve: e.g. $y^2+60=0\Rightarrow y^2=-60\Rightarrow ...$

A1: Fully correct argument that requires:

- fully correct work
- a justification depending on their strategy
- a conclusion depending on their strategy

Via discriminant: $4-4\times1\times16<0$ so no real roots so they do not meet again

Via completing the square: $\rightarrow(x-1)^2+15$ which has a minimum value of 15 so no real roots so they do not meet again.

Via calculus: $x=1\Rightarrow y=15$ is the minimum so no real roots so they do not meet again.

Via solving: $x=1\pm\sqrt{15}\mathrm{i}$ or e.g. $x=1\pm3.87\mathrm{i}$ or e.g. $x=1\pm\sqrt{-15}$ so math error so they do not meet again.

For y it is likely to be more straightforward e.g. $y^2\neq-60$ which cannot be solved so they do not meet again.

Allow equivalent statements for "they do not meet again" e.g. so they only meet once.

(But do **not** condone incorrect statements such as "therefore P does not meet C again")

A minimum justification could be:

$x^2-2x+16=0\rightarrow b^2-4ac=(-2)^2-4\times1\times16$ **ddM1**

$4-4\times1\times16<0$ so no more roots so no more intersections **A1**

Do not allow e.g.

"$x^2-2x+16=0$ gives a math error so they do not meet again"

as there has been no attempt to show why the "math error" occurs – this scores M0A0

Alternative to (c) by showing the cubic is strictly increasing (or decreasing):

M1A1: As in the main scheme then

$$f(x) = x^3 - 3x^2 + 18x - 16 \Rightarrow f'(x) = 3x^2 - 6x + 18$$

$$3x^2 - 6x + 18 = 3(x^2 - 2x + 6) = 3(x-1)^2 + 15$$

$$3(x-1)^2 + 15 > 0 \text{ so f}(x) \text{ is an increasing function}$$

Hence there can only be one intersection (at $x = 1$) so the normal and curve do not intersect again.

dM1: Differentiates their cubic of the form $ax^3 + bx^2 + cx + d = 0 \quad a, b, c, d \neq 0$ to obtain a 3 term quadratic expression with only coefficient errors on the non-constant terms.

ddM1: This mark requires:

- a correct cubic equation in x
- the correct derivative or a multiple of it
- an attempt to show that the quadratic expression is always positive (or negative)

A1: Fully correct concluding argument e.g. that as the derivative is always positive (or always negative) the function is strictly increasing (or decreasing) and therefore there can only be one intersection (at $x = 1$) so the normal and curve do not meet again.

(12 marks)

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